Winter term
2020/2021

# Topics in PDE
# Homework Sheet 2

**Exercise 2.1**
Show the converse in Lemma 1.2.8.

**Exercise 2.2**
Suppose $T : \mathcal{D}(T) \to \mathcal{H}$ is densely defined and closed. Recall the disjoint decomposition $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$ where $\sigma_p(T)$ denotes the set of eigenvalues $\lambda \in \sigma(T)$, i.e., for which injectivity of $T - \lambda$ fails and

$$\sigma_c(T) = \{\lambda \in \sigma(T) \setminus \sigma_p(T) : \overline{\operatorname{ran}(T-\lambda)} = \mathcal{H}\}$$
$$\sigma_r(T) = \{\lambda \in \sigma(T) \setminus \sigma_p(T) : \overline{\operatorname{ran}(T-\lambda)} \neq \mathcal{H}\}$$

being the continuous, respectively the residual spectrum, for which surjectivity of $T - \lambda$ fails. Show that for $\lambda \in \mathbb{C}$, we have the implications

- $\lambda \in \sigma_p(T) \Rightarrow \overline{\lambda} \in \sigma_p(T^*) \cup \sigma_r(T^*)$,
- $\lambda \in \sigma_r(T) \Rightarrow \overline{\lambda} \in \sigma_p(T^*)$, and
- $\lambda \in \sigma_c(T) \Leftrightarrow \overline{\lambda} \in \sigma_c(T^*)$.

In particular, these imply

$$\sigma_r(T) = \{\lambda \in \mathbb{C} \setminus \sigma_p(T) : \overline{\lambda} \in \sigma_p(T^*)\}\,.$$

**Exercise 2.3**
A densely defined operator $A$ in $\mathcal{H}$ is said to be *normal* if $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $\|A\psi\| = \|A^*\psi\|$ for all $\psi \in \mathcal{D}(A)$. Show the following assertions.

1. Normal operators are closed (and thus $A^{**} = A$).
2. Let $A : \mathcal{D}(A) \to \mathcal{H}$ be densely defined and closed. Then the following statements are equivalent.
   (a) $A$ is normal
   (b) $A^*$ is normal
   (c) $AA^* = A^*A$.

**Exercise 2.4**
Let $A \in \mathfrak{B}(\mathcal{H})$ and define the *spectral radius* as

$$r(A) := \limsup_{n\to\infty} \|A^n\|^{1/n}\,.$$

Show that

1. $r(A) \leq \|A^m\|^{1/m} \leq \|A\|$ for all $m \in \mathbb{N}$. Thus $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$.
2. If $A$ is normal then $r(A) = \|A\|$.
3. If $\mathcal{H}$ is complex-valued then $\sigma(A)$ is not empty and there is at least one $z \in \sigma(A)$ with $|z| = r(A)$, i.e., one has $r(A) = \max\{|z| : z \in \sigma(A)\}$.
4. If $A$ is normal in a complex Hilbert space with $\sigma(A) = \{0\}$ then $A = 0$.