Winter term
2020/2021

Topics in PDE
Homework Sheet 1

Exercise 1.1
Let $A : \mathcal{D}(A) \to \mathcal{H}$ be a densely defined operator. Show that

1. A^{**} extends A if A^* is also densely defined.
2. A is bounded if and only if $A^* \in \mathfrak{B}(\mathcal{H})$. In this case one has $\|A^*\| = \|A\|$.
3. A^{**} is the uniquely determined continuous extension of A to the whole $\mathcal{H}$ if A is bounded.

Exercise 1.2

1. Suppose $A : \mathcal{D}(A) \to \mathcal{H}$ is closable and injective. Show the following assertions.
   (a) A^{-1} is closable if and only if $\overline{A}$ is injective. In this case, one has $\overline{A^{-1}} = (\overline{A})^{-1}$.
   (b) If $\overline{A}$ is injective and $(\overline{A})^{-1}$ is continuous, then $\operatorname{ran}(\overline{A}) = \overline{\operatorname{ran}(A)}$.
2. Suppose $A : \mathcal{H} \to \mathcal{H}$ is a bijective operator. Show that
$$\|A^{-1}\|^{-1} = \inf_{\psi \in \mathcal{H}, \|\psi\|=1} \|A\psi\| .$$
3. Show that if A is self-adjoint and injective, then A^{-1} is also self-adjoint.

Exercise 1.3

1. Let $T : \mathcal{D}(T) \to \mathcal{H}$ be densely defined and closed. Show that
$$\rho(T^*) = \{\overline{z} \in \mathbb{C} : z \in \rho(T)\} \quad \text{and} \quad \sigma(T^*) = \{\overline{z} \in \mathbb{C} : z \in \sigma(T)\} .$$
2. Let $S : \mathcal{D}(S) \to \mathcal{H}$ and $T : \mathcal{D}(T) \to \mathcal{H}$ be bijective operators. Show the following assertions.
   (a) If $\mathcal{D}(S) \subseteq \mathcal{D}(T)$, then $T^{-1} - S^{-1} = T^{-1}(S - T)S^{-1}$.
   (b) If $\mathcal{D}(T) \subseteq \mathcal{D}(S)$, then $T^{-1} - S^{-1} = S^{-1}(S - T)T^{-1}$.
   (c) If $\mathcal{D}(T) = \mathcal{D}(S)$, then $T^{-1} - S^{-1} = T^{-1}(S - T)S^{-1} = S^{-1}(S - T)T^{-1}$.
3. Let $T : \mathcal{D}(T) \to \mathcal{H}$ be closed. Show that
$$\|(T - z)^{-1}\| \geq \frac{1}{\operatorname{dist}(z, \sigma(T))} , \quad z \in \rho(T) .$$

Exercise 1.4
Let $A \in \mathfrak{B}(\mathcal{H})$. Show that $\sigma(A) \subseteq \{z \in \mathbb{C} : |z| \leq \|A\|\}$.

Exercise 1.5
Let $A : \mathcal{D}(A) \to \mathcal{H}$ be a densely defined symmetric operator. Show that

1. A has no non-real eigenvalues and
2. eigenvectors of A associated with different eigenvalues are orthogonal to one another.

Remark: This does not imply that two linearly independent eigenvectors to the same eigenvalue are orthogonal. However, it is no restriction to assume that they are since the Gram–Schmidt process gives an orthonormal basis for $\ker(A - \lambda)$.