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RESTRICTIONS OF FOURIER TRANSFORMS TO QUADRATIC SURFACES AND DECAY OF SOLUTIONS OF WAVE EQUATIONS

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§1. Introduction

Let S be a subset of $\mathbb{R}^n$ and $d\mu$ a positive measure supported on S and of temperate growth at infinity. We consider the following two problems:

Problem A. For which values of p, $1 \leq p < 2$, is it true that $f \in L^p(\mathbb{R}^n)$ implies $\hat{f}$ has a well-defined restriction to S in $L^2(d\mu)$ with

$$\left(\int |\hat{f}|^2 d\mu\right)^{1/2} \leq c_p \|f\|_p? \tag{1.1}$$

Problem B. For which values of q, $2 < q \leq \infty$, is it true that the tempered distribution $Fd\mu$ for each $F \in L^2(d\mu)$ has Fourier transform in $L^q(\mathbb{R}^n)$ with

$$\|(Fd\mu)^{\wedge}\|_q \leq c_q\left(\int |F|^2 d\mu\right)^{1/2}? \tag{1.2}$$

A simple duality argument shows these two problems are completely equivalent if p and q are dual indices, $(1/p) + (1/q) = 1$. Interest in Problem A when S is a sphere stems from the work of C. Fefferman [3], and in this case the answer is known (see [11]). Interest in Problem B was recently signalled by I. Segal [6] who studied the special case $S = \{(x, y) \in \mathbb{R}^2 : y^2 - x^2 = 1\}$ and gave the interpretation of the answer as a space-time decay for solutions of the Klein-Gordon equation with finite relativistic-invariant norm.

In this paper we give a complete solution when S is a quadratic surface given by

$$S = \{x \in \mathbb{R}^n : R(x) = r\} \tag{1.3}$$

where $R(x)$ is a polynomial of degree two with real coefficients and r is a real constant. To avoid triviality we assume R is not a function of fewer than n variables, so that aside from isolated points S is a $n - 1$-dimensional C^∞ manifold. There is a canonical measure $d\mu$ associated to the function R (not intrinsic to the surface S, however) given by

$$d\mu = \frac{dx_1 \cdots dx_{n-1}}{|\partial R/\partial x_n|} \tag{1.4}$$

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in any neighborhood in which $\dfrac{\partial R}{\partial x_n} \neq 0$ so that S may be described by giving x_n as a function of $x_1, \cdots x_{n-1}$.

In outline our solution follows closely the argument in the special case of the sphere. We begin with an observation due to P. Tomas [11]:

LEMMA 1. *Suppose we can prove*

$$\|(d\mu)^{\wedge} * g\|_q \leq c_p^2 \|g\|_p \tag{1.5}$$

for some p, $1 \leq p < 2$, $(1/q) + (1/p) = 1$. *Then Problem A is true for that* p, *and the same constant in* (1.5) *works in* (1.1).

Proof. Observe $\int \bar{\hat{f}} \hat{f} d\mu = \int \bar{f} (\hat{f} d\mu)^{\vee} = \int \bar{f} ((d\mu)^{\wedge} * f)$ so

$$\int |\hat{f}|^2 d\mu \leq \|f\|_p \|(d\mu)^{\wedge} * f\|_q$$
$$\leq c_p^2 \|f\|_p^2 .$$

Next we use an interpolation idea due to E. M. Stein (unpublished). We consider the analytic family of generalized function $G_z(x) = \gamma(z)(R(x) - r)_+^z$ for an appropriate analytic function $\gamma(z)$ with a simple zero at $z = -1$. From the definition of $d\mu_t$ (now we let $r = t$ vary) we have

$$\int G_z(x)\varphi(x)dx = \gamma(z) \int \left(\int_{S_t} \varphi d\mu_t \right) (t - r)_+^z dt$$

so it follows from the one-dimensional analysis of $(t - r)_+^z$(see [4]) that

$$\lim_{z \to -1} \int G_z(x)\varphi(x)dx = c \int_S \varphi d\mu .$$

By analytic continuation we may extend G_z to the strip $-\lambda_0 \leq \mathrm{Re}(z) \leq 0$ and consider the associated analytic family of operators $T_z g = \mathcal{F}^{-1}(G_z \hat{g}) = \check{G}_z * g$. Because G_z is bounded on $\mathrm{Re}(z) = 0$ we have

$$\|T_z g\|_2 \leq |\gamma(z)| \, \|g\|_2$$

on this line. We will always choose $\gamma(z)$ so that $|\gamma(it)|$ has at most exponential growth at infinity. As an immediate application of Stein's interpolation theorem (see [7]) we have

LEMMA 2. *Suppose we can prove* $\check{G}_z$ *is bounded for* $\mathrm{Re}(z) = -\lambda_0$ *with* $\lambda_0 > 1$ *and* $\|\check{G}_{-\lambda_0 + it}\|_\infty$ *has at most exponential growth at infinity. Then* (1.5) *holds for*

$$p = \frac{2\lambda_0}{\lambda_0 + 1}, \qquad q = \frac{2\lambda_0}{\lambda_0 - 1} .$$

The problem is then reduced to the explicit computation of $\check{G}_z$ from which we can read off the values of z for which it is bounded. We do this in a case-by-case manner in §2. We show that in most cases the value $\lambda_0 = \dfrac{n+1}{2}$ will work (this yields $p = 2(n+1)/(n+3)$), with the exception of the cone $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_m = 0$ (and equivalent surfaces) for which we must take $\lambda_0 = (n/2)$ hence $p = 2n/(n+2)$.

To show that our results are best possible, in the sense that we cannot increase p or decrease q, we use the following idea due to A. W. Knapp (unpublished):

LEMMA 3. *Let S be an $n-1$-dimensional C^∞ manifold embedded in $\mathbb{R}^n$ and $d\mu$ any smooth non-vanishing measure on S. Then* (1.1) *cannot hold unless* $p \le 2(n+1)/(n+3)$.

Proof. Consider a rectangular region in $\mathbb{R}^n$ with one side of length ϵ and $n-1$ sides of length $\sqrt{\epsilon}$. Move this region so that it is centered at a point of S and rotate it so the side of length ϵ lies normal to S. The characteristic function of this region will be $\hat{f}$ in (1.1). Then S intersects the region in a set which is roughly an $n-1$-dimensional square with sides of length $\sqrt{\epsilon}$, so $\int |\hat{f}|^2 d\mu \ge c\epsilon^{(n-1)/2}$ as $\epsilon \to 0$. On the other hand, ignoring inessentials, $f(x)$ is $\dfrac{\sin \epsilon x_1}{x_1} \dfrac{\sin \sqrt{\epsilon}\, x_2}{x_2} \cdots \dfrac{\sin \sqrt{\epsilon}\, x_n}{x_n}$ so a direct computation shows $\|f\|_p^p = c\,\epsilon^{p-1} \cdot \epsilon^{(n-1)(p-1)/2}$. Thus (1.1) would imply $\epsilon^{(n-1)/4} \le c\epsilon^{(n+1)/2q}$ as $\epsilon \to 0$ hence $(n-1)/4 \ge (n+1)/2q$ which is equivalent to $p \le 2(n+1)/(n+3)$.

The argument can be modified in the case of the cone by taking the side of the rectangular region lying in the radial direction (the direction in which the cone is flat) to have length one. Then $\int |\hat{f}|^2 d\mu \ge c\epsilon^{(n-2)/2}$ while $\|f\|^p{}_p \le c\epsilon^{p-1}$ $\epsilon^{(n-2)(p-1)/2}$ so the same argument yields $p \le 2n/(n+2)$.

This completes the outline of our methods. The exact statement of results is given in Theorem 1 in section 2. We give some applications to decay of solutions of wave equations in section 3, including the generalization to higher dimensions of results of Segal [6] (actually we have a slight improvement of Segal's result even in one space dimension).

For unexplained notation the reader should consult [4].

We are grateful to Professor A. W. Knapp for explaining his unpublished work and for permission to present it here.

Finally we want to point out that none of the methods used in this paper are new; even the computations in section 2 can be found in some form in [4].

§2. Computations of Fourier transforms

In this section we discuss three special cases:

Case I. $S = \{x_n - Q'(x', x') = 0\}$ where $x' = (x_1, \cdots, x_{n-1})$, Q' is a non-degenerate quadratic form on $\mathbb{R}^{n-1}$ of signature a, b

$$Q'(x', x') = x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - \cdots - x_{n-1}^2, \tag{2.1}$$

where a, b are non-negative integers such that $a + b = n - 1$.

Case II. $S = \{Q(x, x) = 0\}$ where Q is a non-degenerate quadratic form on $\mathbb{R}^n$ of signature a, b

$$Q(x, x) = x_1^2 + \cdots + x_a^2 - x_{a+1}^2 \cdots - x_n^2, \tag{2.2}$$

where a, b are positive integers such that $a + b = n$.

Case III. $S = \{Q(x, x) = 1\}$ where Q is of the form (2.2) and a, b satisfy $a + b = n$, $a \neq 0$.

It is a simple exercise in algebra to show that any quadratic surface not contained in an affine hyperplane can be transformed into one of these three cases under an affine transformation. Since affine transformations don't influence the solution of problems A and B, it suffices to consider just these three cases.

Case I. We let

$$G_z(x) = \Gamma(z + 1)^{-1}(x_n - Q'(x', x'))_+^z.$$

The computation of the inverse Fourier transform of G_z is essentially well-known. We give a formal outline:

First we begin with

$$\int_0^\infty x^z e^{-ixy} dx = ie^{iz\pi/2}\Gamma(z + 1)(-y + i0)^{-z-1}$$

(see [43], p. 360) from which we obtain

$$\Gamma(z + 1)^{-1} \int_{-\infty}^\infty e^{-ix_n y_n}(x_n - Q'(x', x'))_+^z \, dx_n = ie^{iz\pi/2}e^{iy_n Q'(x',x')}(-y_n + i0)^{-z-1}. \tag{2.3}$$

Next we use

$$\int_{-\infty}^\infty e^{itx^2}e^{ixy}dx = \sqrt{4\pi}\ |t|^{-1/2}e^{-\frac{\pi i}{4}\operatorname{sgn} t}e^{-iy^2/4t}.$$

Combining this with (2.3) we obtain

$$(2\pi)^{-n}\Gamma(z + 1)^{-1} \int_{\mathbb{R}^n} e^{-ix \cdot y}(x_n - Q'(x', x'))^z dx$$

$$= \pi^{-\frac{n}{2}} i\, e^{iz\pi/2}e^{(\pi i/4)(b - a)}e^{-iQ'(y',y')/4t}$$

$$|y_n|^{-(n-1)/2}(-y_n + i0)^{-z-1}.$$

This is clearly bounded if and only if $\mathrm{Re}(-z - 1 - (n-1)/2) = 0$, in other words $\mathrm{Re}(z) = -(n+1)/2$ and the growth as $\mathrm{Im}(z) \to 0$ is exponential.

Case II. We let $n \geq 3$ and $G_z(x) = \Gamma(z+1)^{-1}\Gamma(z + (n/2))^{-1}Q(x, x)_+^z$ (if $n = 2$ then S is a pair of straight lines). The computation of the Fourier transform may be found in Gelfand-Shilov [4], p. 365:

$$\text{(2.4)} \qquad \hat{G}_z(y) = 2^{n+2z}\pi^{(n/2)-1}(2i)^{-1}[e^{-i(z+(b/2))\pi}\cdot (Q(y, y) - i0)^{-z-(n/2)} - e^{i(z+(b/2))\pi}(Q(y, y) + i0)^{-z-(n/2)}].$$

It is clear that we have the appropriate boundedness if and only if $\mathrm{Re}(z) = -n/2$.

Case III. We let

$$G_z(x) = h(z)\Gamma(z+1)^{-1}(1 - Q(x, x))_+^z$$

where $h(z)$ will be specified later. The computation of $\hat{G}_z$ is also given in Gelfand-Shilov [4] p. 290 (an alternate derivation may be given following a similar computation in [10] p. 516).

$$\text{(2.5)} \quad \hat{G}_z(y) = h(z)2^{z+(n/2)+1}\pi^{(n/2)-1}\cdot \left\{-\sin\pi(z + (a/2))\frac{K_{z+(n/2)}(Q(y, y)_-^{1/2})}{Q(y, y)_-^{(1/2)(z+(n/2))}}\right. + \frac{\pi}{2\sin\pi(z + (n/2))}\left[\sin\pi(z + a/2)\frac{J_{z+(n/2)}(Q(y, y)_+^{1/2})}{Q(y, y)_+^{(1/2)(z+(n/2))}}\right. \left.\left. + \sin\pi b/2\,\frac{J_{-z-(n/2)}(Q(y, y)_+^{1/2})}{Q(y, y)_+^{(1/2)(z+(n/2))}}\right]\right\}.$$

Note that if Q is positive definite ($b = 0$) the first and third term are zero and (2.5) reduces to the well-known formula for $((1 - |x|^2)_+^z)^\wedge$. Taking $h(z) = 1$ we obtain boundedness for $\mathrm{Re}(z) \geq -(n+1)/2$.

Next consider the indefinite case ($a \neq 0$, $b \neq 0$) and $n \geq 3$. Choose $h(z) = (z + (n/2))\sin\pi(z + (n/2))$ if n is odd and $h(z) = (z + (n/2))\cdot(z+1)^{-1}\sin\pi(z + (n/2))$ if n is even. Note that $h(-1) \neq 0$ so G_{-1} is a non-zero multiple of $d\mu$. We claim $\hat{G}_z$ is bounded if $-n/2 \geq \mathrm{Re}(z) \geq -(n+1)/2$. Note that the factor $h(z)$ cancels the poles of $(\sin\pi(z + (n/2)))^{-1}$ in this region (here we use $n \neq 2$) so by setting $\lambda = -z - (n/2)$ and $u = Q(y, y)_\pm^{1/2}$ the problem reduces to the boundedness of $u^\lambda J_\lambda(u)$, $u^\lambda J_{-\lambda}(u)$ and $\lambda u^\lambda K_\lambda(u)$ for $0 \leq \mathrm{Re}\lambda \leq 1/2$. For the first two terms this follows from the well-known estimates

$$J_\lambda(u) = \begin{cases} 0(u^\lambda) & u \to 0 \\ 0(u^{-1/2}) & u \to \infty \end{cases}$$

for the Bessel functions. For the last term we use the integral formula

$K_\lambda(u) = \int_0^\infty \cosh \lambda t \, e^{-u \cos ht} dt$ to obtain exponential decay as $u \to \infty$ ([12], p. 259) and the power series expansion ([12], p. 270)

$$\lambda u^\lambda K_\lambda(u) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot$$

$$[\lambda\Gamma(-k - \lambda)(u/2)^{2\lambda} + \lambda\Gamma(\lambda - k)](u/2)^{2k}$$

to obtain boundedness for small u. Notice that the factor λ cancels the poles of the Γ-function at $\lambda = 0$.

Finally we consider the remaining case $n = 2$, $a = b = 1$. Here we choose $h(z) = (z + 1)^{-1} \sin \pi(z + 1)$. Here we have boundedness only for

$$-1 > \mathrm{Re}(z) \geq -3/2$$

since when $z = -1$ the K term has a pole. This completes the proof of the affirmative part of

THEOREM 1. *A necessary and sufficient condition for the affirmative solution to Problems A and B is*

Case I. $p = 2(n + 1)/(n + 3)$
$q = 2(n + 1)/(n - 1)$.

Case II. $n \geq 3$ and
$p = 2n/(n + 2)$
$q = 2n/(n - 2)$.

Case III. (a) $a = n, b = 0$ and
$1 \leq p \leq 2(n + 1)/(n + 3)$
$2(n + 1)/(n - 1) \leq q \leq \infty$.
(b) $a \neq 0, b \neq 0, n \geq 3$ and
$2n/(n + 2) \leq p \leq 2(n + 1)/(n + 3)$
$2(n + 1)/(n - 1) \leq q \leq 2n/(n - 2)$.
(c) $a = b = 1, n = 2$ and
$1 < p \leq 6/5$
$6 \leq q < \infty$.

Proof. The sufficiency follows from Lemma 2 and the above computations. Furthermore Lemma 3 shows that upper bounds for p are necessary. The necessity of the lower bounds for p follows from elementary considerations involving the growth of $d\mu$ at infinity.

Consider Case III(b). Choose

$$\hat{f}(y) = (1 + |y|^2)^{(\alpha - n)/2}.$$

Then $f \in L^p$ provided $\alpha p < n$ (f has a singularity like $|x|^{-\alpha}$ near zero and exponential decay at infinity; see [7]). But

$$\int |\hat{f}(y)|^2 d\mu(y) = \int |\hat{f}(y)|^2 \frac{dy_1 \cdots dy_{n-1}}{|y_n|}.$$

By considering only large values of y for which $|y_n| \geq |y|/2$ we can bound this from below by a multiple of

$$\int_{\mathbb{R}^{n-1}} (1 + |y'|^2)^{\alpha - n - 1/2} dy'$$

which diverges if $\alpha = (n + 2)/2$. Thus for any $p < 2n/(n + 2)$ we have $f \in L^p$ but $\hat{f} \notin L^2(d\mu)$, proving the necessity of $p \geq 2n/(n + 2)$. The identical argument works for case II. There is nothing to prove in case III(a). In case III(c) we have to show that (1.1) cannot hold for $p = 1$. To do this consider $\hat{f}(y) = e^{-t|y|^2}$. Note that $\|f\|_1$ is independent of t, so (1.1) and the monotone convergence theorem would imply $1 \in L^2(d\mu)$ which is false since $d\mu$ is an infinite measure.

Finally for case I we use a homogeneity argument with respect to the non-isotropic dilations

$$\delta_t f(x) = f(tx_1, \cdots, tx_{n-1}, t^2 x_n).$$

Note that $(\delta_t f)\hat{\ }(y) = t^{-n-1}\delta_{t^{-1}}\hat{f}(y)$. Now $\|\delta_t f\|_p = t^{-(n+1)/p}\|f\|_p$ while

$$\|t^{-n-1}\delta_{t^{-1}}\hat{f}(y)\|_{L^2(d\mu)} = t^{-n-1}\left(\int_{\mathbb{R}^{n-1}} \hat{f}(t^{-1}y', t^{-2}|y'|^2)dy'\right)^{1/2}$$

$$= t^{-(n+3)/2}\|f\|_{L^2(d\mu)}.$$

Thus for (1.1) to hold we must equate the two powers of t, hence $p = 2(n + 1)/(n + 3)$.

§3. Estimates for solutions of wave equations

It is a simple matter to interpret to the solution to Problem B as a statement about solutions of partial differential equations. We consider three "wave equations" of mathematical physics, the free Schrödinger equation, the Klein-Gordon equation, and the acoustic wave equation.

COROLLARY 1. *Let $u(x, t)$ be a solution of the inhomogeneous free Schrödinger equation in n-space dimensions*

$$i\frac{\partial u}{\partial t}(x, t) + \lambda\Delta_x u(x, t) = g(x, t) \tag{3.1}$$

for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, λ a non-zero real constant, with initial data

$$u(x, 0) = f(x). \tag{3.2}$$

Assume $f \in L^2(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^{n+1})$ for $p = 2(n + 2)/(n + 4)$. Then $u \in L^q(\mathbb{R}^{n+1})$ for $q = 2(n + 2)/n$ and $\|u\|_q \leq c(\|f\|_2 + \|g\|_p)$.

Proof. It is well-known that (3.1) and (3.2) have a unique solution which may be written $u(x, t) = \int_0^t e^{i\lambda(t-s)\Delta} g(\cdot, s)ds + c\int_{\mathbb{R}^n} \hat{f}(y)e^{-i(x\cdot y + \lambda|y|^2 t)}dy$. The estimate for the second term is an immediate consequence of Theorem 1 (Problem B case I $R(x, t) = t - \lambda|x|^2$ in $n + 1$ dimensions). To estimate the first term observe $e^{it\Delta}h(x) = ct^{-n/2}\int_{\mathbb{R}^n} e^{i|x-y|^2/4t}h(y)dy$ hence $\|e^{it\Delta}h\|_\infty \leq c|t|^{-n/2}\|h\|_1$, while also $\|e^{it\Delta}h\|_2 = \|h\|_2$. Interpolating these inequalities we obtain $\|e^{it\Delta}h\|_q \leq c|t|^{-r}\|h\|_p$ for $r = n(p^{-1} - 2^{-1}) = n/(n + 2)$. Then $\left\|\int_0^t e^{i\lambda(t-s)\Delta}g(\cdot, s)ds\right\|_{L^q(\mathbb{R}^n)} \leq c\int_0^t |t - s|^{-r}\|g(\cdot, s)\|_{L^p(\mathbb{R}^n)}ds$. We may apply the fractional integration theorem to obtain $\left\|\int_0^t e^{i\lambda(t-s)\Delta}g(\cdot, s)ds\right\|_{L^q(\mathbb{R}^{n+1})} \leq c\|g\|_p$ because $p^{-1} - q^{-1} = 1 - r = 2/(n + 2)$.

Remark. It would be interesting to extend the result to Schrödinger's equation with potential (for suitable $V(x)$) $i\dfrac{\partial u}{\partial t} + \lambda\Delta_x u = V(x)u$. It is not hard to modify the argument given in the proof of theorem 1 to handle this equation provided the key estimate

$$\|e^{it(\Delta + V)}h\|_\infty \leq c|t|^{-n/2}\|h\|_1$$

can be established. The Feynman integral representation of the operator $e^{it(\Delta + V)}$ (see [1]) is very suggestive of such an estimate.

COROLLARY 2. (cf. Segal [6]) *Let $u(x, t)$ be a solution of the Klein-Gordon ($m > 0$) or d'Alembertian wave equation ($m = 0$)*

$$-\frac{\partial^2 u}{\partial t^2}(x, t) + \Delta_x u(x, t) - m^2 u(x, t) = g(x, t) \tag{3.3}$$

for $x \in \mathbb{R}^n$, $t \in \mathbb{R}^1$ with Cauchy data

$$u(x, 0) = f_0(x), \qquad \frac{\partial u}{\partial t}(x, 0) = f_1(x). \tag{3.4}$$

Assume $B^{1/2}f_0 \in L^2(\mathbb{R}^n)$ and $B^{-1/2}f_1 \in L^2(\mathbb{R}^n)$ where $B = (m^2 - \Delta)^{1/2}$ and $g \in L^p(\mathbb{R}^{n+1})$ (see below for restrictions on p). Then $u \in L^q(\mathbb{R}^{n+1})$ and $\|u\|_q \leq c(\|B^{1/2}f_0\|_2 + \|B^{-1/2}f_1\|_2 + \|g\|_p)$. Here $p^{-1} + q^{-1} = 1$ and

(a) *If $m > 0$ we must have*

$$2(n + 1)/(n + 3) \leq p \leq 2(n + 2)/(n + 4)$$

$$2(n + 2)/n \leq q \leq 2(n + 1)/(n - 1)$$

for $n \geq 2$ and

$$1 < p \leq 6/5, \qquad 6 \leq q < \infty \quad \text{for} \quad n = 1;$$

(b) *if* $m = 0$ *we must have* $n \geq 2$ *and*

$$p = 2(n + 1)/(n + 3)$$

$$q = 2(n + 1)/(n - 1).$$

Also in this case we must subtract a suitable constant from u (or else assume that f_0 *vanishes at infinity in a suitable sense.)*

Proof. Consider the homogeneous equation $g \equiv 0$. It is well-known that (3.3) and (3.4) have a unique solution which may be written (aside from a constant in the case $m = 0$)

$$(3.5) \qquad u(x, t) = \int_{\mathbb{R}^n} e^{-i(x \cdot y + \sqrt{m^2 + |y|^2}\, t)} \varphi_+(y)dy \Big/ \sqrt{m^2 + |y|^2}$$
$$+ \int_{\mathbb{R}^n} e^{-i(x \cdot y - \sqrt{m^2 + |y|^2}\, t)} \varphi_+(y)dy \Big/ \sqrt{m^2 + |y|^2} \quad ,$$

where $\varphi_t = (1/2)(\widehat{Bf_0} + i\hat{f}_1)$ and $\varphi_- = (1/2)(\widehat{Bf_0} - i\hat{f}_1)$. Thus $u = \mathscr{F}^{-1}(Fd\mu)$ where $d\mu$ is the measure (1.4) on the two-sheeted hyperboloid ($m > 0$) or cone ($m = 0$) $-|x|^2 + t^2 = m^2$ in $n + 1$ dimensions and $F = \varphi\pm$ on each sheet. Note that

$$\|F\|^2_{L^2(d\mu)} = \int (|\varphi_+(y)|^2 + |\varphi_-(y)|^2)dy \Big/ \sqrt{m^2 + |y|^2}$$
$$= \|B^{1/2}f_0\|_2^2 + \|B^{-1/2}f_1\|_2^2$$

by the Plancherel formula. Thus the desired estimate follows from theorem 1 (replacing n by $n + 1$) case II and case III.

Finally, for the inhomogeneous equation with zero Cauchy data the desired estimates are already known, see [9] for $m = 0$ and [10] for $m > 0$ (note that the arguments on p. 515–517 of [10] actually prove the result for the values of p and q given here, even though the result is only stated for a more restricted set of values). An arbitrary solution of (3.3) and (3.4) is a sum of two solutions of the special kinds considered.

Remark. The conditions on the Cauchy data $B^{1/2}f_0 \in L^2$ and $B^{-1/2}f_1 \in L^2$ may be considered natural because (at least for $g \equiv 0$) they are Lorentz-invariant as well as being time independent. However for some problems it may be convenient to use other norms on the Cauchy data which are time independent, such as

$$\|(f_0, f_1)\|^2_{(s)} = \|B^{(s+1)/2}f_0\|_2^2 + \|B^{(s-1)/2}f_1\|_2^2.$$

For example, when $s = 1$ the norm squared has the interpretation of energy. Now it is an easy matter to see from the representation (3.5) that a solution of

$$\frac{\partial^2 u}{\partial t^2} + \Delta_x u - m^2 u = 0$$

with Cauchy data with finite s-norm is obtained from a solution of the same equation with finite 0-norm by applying a Bessel potential ($m > 0$) or Riesz potential ($m = 0$) of order s. Thus $u \in L_s^q(\mathbb{R}^{n+1})$ when $m > 0$, the usual Sobolev space of Bessel potentials (see [7]), and $u \in \mathring{L}_s^q(\mathbb{R}^{n+1})$ when $m = 0$, the homogeneous Sobolev space of Riesz potentials (say for $0 < s < n/q$ to avoid technicalities). Applying the fractional integration theorem (or Sobolev's inequality) we obtain the estimate

$$\|u\|_q \leq c\|(f_0, f_1)\|_{(s)}$$

for $s \geq 0$ and

(a) $$2(n+2)/n \leq q \leq 2(n+1)/(n-1-2s) \quad \text{if} \quad s < (n-1)/2$$

$$2(n+2)/n \leq q < \infty \quad \text{if} \quad s = (n-1)/2$$

$$2(n+2)/n \leq q \leq \infty \quad \text{if} \quad s > (n-1)/2 \quad \text{for} \quad m > 0$$

(b) $$q = 2(n+1)/(n-1-2s) \quad \text{if} \quad 0 \leq s < (n-1)/2 \quad \text{for} \quad m = 0.$$

These results are best possible when $s \geq 0$ as can be seen by simple modification of arguments already given. Also when $m = 0$ there can be no such estimates for $s < 0$ since a homogeneity argument would require $q = 2(n+1)/(n+1-2s)$ which contradicts the proof of Lemma 3. We leave open the possibility of such estimates when $m > 0$ and $0 > s > -1/(n+2)$.

References

1. S. A. Albeverio and R. J. Høegh-Krohn, *Mathematical Theory of Feynman Path Integrals*, Springer-Verlag, 1976.
2. L. Carleson and P. Sjölin, *Oscillatory integrals and a multiplier problem for the disc*, Studia Math. **44**(1972), 287–299.
3. C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124**(1970), 9–36.
4. I. M. Gelfand and G. E. Shilov, *Generalized Functions*, vol. I, Academic Press, New York, 1964.
5. L. Hörmander, *Oscillatory integrals and multipliers on FL^p*, Ark. Mat. **11**(1973), 1–11.
6. I. Segal, *Space-time decay for solutions of wave equations*, Advances in Math. **22**(1976), 305–311.
7. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, N.J., 1970.
8. R. Strichartz, *Convolutions with kernels having singularities on a sphere*, Trans. Amer. Math. Soc. **148**(1970), 461–471.
9. ______, *A priori estimates for the wave equation and some applications*, J. Functional Analysis **5**(1970), 218–235.
10. ______, *Fourier transforms and non-compact rotation groups*, Indiana Univ. Math. J. **24**(1974), 499–525.
11. P. Tomas, *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc. **81**(1975), 477–478.
12. N. J. Vilenkin, *Special Functions and the Theory of Group Representations*, Translations of Mathematical Monographs, vol. 22, American Mathematical Society, 1968.

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