

Fourier Restriction and Applications

Homework Sheet 5

Exercise 5.1

Suppose $\varphi \in C^\infty((a, b) : \mathbb{R})$ with $\varphi'(x) \neq 0$ and $\psi \in C^\infty((a, b) : \mathbb{C})$ not necessarily vanishing at a and b . Find a condition on φ and ψ such that for all $N \in \mathbb{N}$, one has $|\int_a^b e^{i\lambda\varphi(x)}\psi(x) dx| \lesssim_{\varphi,\psi,a,b,N} \lambda^{-N}$.

Exercise 5.2

Find a counterexample that shows that a simple lower bound for $|\varphi'(x)|$ alone does not suffice for van der Corput's lemma (Proposition 3.1.2) to hold.

Exercise 5.3

Prove Corollary 3.1.3 in the notes.

Exercise 5.4

1. Let $\ell \in \mathbb{N}_0$. Show that

$$I_\ell(\lambda) = \int_{\mathbb{R}} e^{i\lambda x^2} x^\ell e^{-x^2} dx \sim \lambda^{-\frac{\ell+1}{2}} \sum_{j \geq 0} c_j^{(\ell)} \lambda^{-j} \quad \text{as } \lambda \rightarrow \infty \quad (1)$$

for certain $\{c_j^{(\ell)}\}_{j \in \mathbb{N}_0} \subseteq \mathbb{C}$, i.e., for all $N, r \in \mathbb{N}_0$ one has

$$\left| \left(\frac{d}{d\lambda} \right)^r \left[I_\ell(\lambda) - \lambda^{-\frac{\ell+1}{2}} \sum_{j=0}^N c_j^{(\ell)} \lambda^{-j} \right] \right| \lesssim \lambda^{-r - (\frac{\ell+1}{2} + N + 1)} \quad \text{as } \lambda \rightarrow \infty.$$

2. Let $\eta \in C_c^\infty$, $g \in \mathcal{S}(\mathbb{R})$ with $g(x) = 0$ for $|x|$ sufficiently small, and $\ell, N \in \mathbb{N}_0$. Suppose we knew the estimates

$$\left| \int_{\mathbb{R}} e^{i\lambda x^2} x^\ell \eta(x) dx \right| \lesssim_{\eta, \ell} \lambda^{-\frac{\ell+1}{2}} \quad (2)$$

and

$$\left| \int_{\mathbb{R}} e^{i\lambda x^2} g(x) dx \right| \lesssim_{g, N} \lambda^{-N}. \quad (3)$$

Suppose $\psi \in C_c^\infty(\mathbb{R})$ is supported in a sufficiently small neighborhood around $x = 0$. Combine (1), (2), and (3) to show

$$\int_{\mathbb{R}} e^{i\lambda x^2} \psi(x) dx \sim \lambda^{-1/2} \sum_{j \geq 0} a_j \lambda^{-j/2} \quad \text{as } \lambda \rightarrow \infty$$

for certain $\{a_j\}_{j \in \mathbb{N}_0} \subseteq \mathbb{C}$.

Hint: Multiply and divide the integrand on the left side by e^{-x^2} , write $\psi(x) = \tilde{\psi}(x)\psi(x)$ for some $\tilde{\psi} \in C_c^\infty(\mathbb{R})$ with $\tilde{\psi}(x) = 1$ for $x \in \text{supp}(\psi)$, and expand $e^{x^2}\psi(x)$ in a Taylor series.