Summer term 2025,
to be discussed on
May 27, 2025

Fourier Restriction and Applications
Homework Sheet 3

Exercise 3.1
Let $0 < p \le \infty$, $1 < q \le \infty$ and assume (X, μ) and (Y, ν) are two measure spaces. Let T be a sublinear operator (initially defined on the set of really simple functions $f = \sum_{k=1}^N a_k \mathbf{1}_{E_k}$ on X such that Tf is a ν-measurable function on Y), i.e., $|T(f+g)| \le |Tf| + |Tg|$ and $|T(\lambda f)| = |\lambda||Tf|$ for $f, g \in \operatorname{dom}(T)$ and $\lambda \in \mathbb{C}$. We say that T is of restricted weak type (p, q) if

$$\alpha d_{T\mathbf{1}_E}(\alpha)^{1/q} \lesssim |E|^{1/p} \quad \text{for all } \alpha > 0\,, E \subseteq X\,.$$

Prove that T is of restricted weak type (p, q) if and only if

$$|\int_F (T\mathbf{1}_E)(x)\, d\nu(x)| =: |\langle \mathbf{1}_F, T\mathbf{1}_E\rangle| \lesssim |E|^{1/p}|F|^{1/q'}$$

for all $E \subseteq X$ and $F \subseteq Y$. (Hint: Use Proposition 1.2.18 for "$\Leftarrow$" and Hölder's inequality or the layer-cake representation and Fubini to prove "$\Rightarrow$".)

Exercise 3.2
In Lemma 1.3.5, show that the sub-double exponential hypothesis $|f(z)| \lesssim_f \exp(\mathcal{O}_f(\mathrm{e}^{(\pi-\delta)|z|}))$ for some $\delta > 0$ is completely sharp.

Exercise 3.3 (Hadamard's three circle theorem)
Let $g(z)$ be holomorphic on the annulus $A := \{z \in \mathbb{C} : \; R_1 \le |z| \le R_3\}$ for some $0 < R_1 < R_3$ and denote

$$M(r) := \max_{\theta \in [0, 2\pi]} |g(R\mathrm{e}^{i\theta})| \quad \text{for } R \in (R_1, R_3).$$

Prove that

$$\log\left(\frac{R_3}{R_1}\right) \log M(R_2) \le \log\left(\frac{R_3}{R_2}\right) \log M(R_1) + \log\left(\frac{R_2}{R_1}\right) \log M(R_3)$$

for any $R_1 < R_2 < R_3$, i.e., $\log M(R)$ is a convex function of $\log R$.
(Hint: Convince yourself that there is a vertical strip which the exponential function maps onto the annulus A.)
Compute both sides of the inequality when $g(z) = cz^\lambda$ for some constants $c \in \mathbb{C}$ and $\lambda \in \mathbb{Z}$.