Summer term 2025,
to be discussed on
May 13, 2025

Fourier Restriction and Applications
Homework Sheet 2

Exercise 2.1
Show the following

Lemma 0.1. *Let* $(X, \|\cdot\|)$ *be a quasi-normed space, i.e.,* $\|f+g\| \leq c_1(\|f\| + \|g\|)$ *for some* $c_1 \geq 1$. *Assume that a sequence* $(f_k)_{k\in\mathbb{N}} \in X$ *satisfies* $\|f_k\| \lesssim A \cdot c_2^{-k}$ *for some* $A > 0$ *and* $c_2 > 1$. *Then* $\|\sum_{k=1}^N f_k\| \leq A \cdot c_3$ *where* c_3 *does not depend on* A *or* N *(but possibly on* c_1 *and* c_2*).*

(Why is this assertion non-trivial?)

Exercise 2.2
Let $0 < p \leq \infty$, $1 < q \leq \infty$ and assume (X, μ) and (Y, ν) are two measure spaces. Let T be a sublinear operator (initially defined on the set of really simple functions $f = \sum_{k=1}^N a_k \mathbf{1}_{E_k}$ on X such that Tf is a ν-measurable function on Y), i.e., $|T(f+g)| \leq |Tf| + |Tg|$ and $|T(\lambda f)| = |\lambda||Tf|$ for $f, g \in \operatorname{dom}(T)$ and $\lambda \in \mathbb{C}$. We say that T is of restricted weak type (p, q) if

$$\alpha d_{T\mathbf{1}_E}(\alpha)^{1/q} \lesssim |E|^{1/p} \quad \text{for all } \alpha > 0\,, E \subseteq X\,.$$

Prove that T is of restricted weak type (p, q) if and only if

$$\left|\int_F (T\mathbf{1}_E)(x)\, d\nu(x)\right| =: |\langle \mathbf{1}_F, T\mathbf{1}_E\rangle| \lesssim |E|^{1/p}|F|^{1/q'}$$

for all $E \subseteq X$ and $F \subseteq Y$. (Hint: Use Proposition 1.2.18 for "$\Leftarrow$" and Hölder's inequality or the layer-cake representation and Fubini to prove "$\Rightarrow$".)

Exercise 2.3
Let $A, B_0, B_1 \geq 0$ such that $A \leq \min(B_0, B_1)$. Show that for all $\theta \in (0,1)$ we have

$$A \leq B_\theta \min\left(\frac{B_0}{B_1}, \frac{B_1}{B_0}\right)^\varepsilon$$

for any sufficiently small $\varepsilon_\theta > 0$, where $B_\theta = B_0^{1-\theta} B_1^\theta$. Can you imagine a situation where this type of estimate is superior to the simpler estimate $A \leq B_\theta$? (For hints, see `https://terrytao.wordpress.com/2009/03/30/245c-notes-1-interpolation-of-lp-spaces/`.)