Summer term 2025,
to be discussed on
April 22, 2025

Fourier Restriction and Applications
Homework Sheet 1

Exercise 1.1
Let $0 < p < 1$ and assume $f, (f_k)_{k\in\mathbb{N}}, g \in L^p(X,\mu)$.

1. Show that
$$\| \sum_{k=1}^{N} f_k \|_{L^p(X)} \leq N^{1/p-1} \sum_{k=1}^{N} \|f_k\|_{L^p(X)}$$
holds for any $N \in \mathbb{N}$. (Hint: First show $\| \sum_{k=1}^{N} f_k\|_p^p \leq \sum_{k=1}^{N} \|f_k\|_p^p$ and then use Hölder's inequality for the sum on the right side.) Is the constant $N^{1/p-1}$ sharp?

2. If additionally $f, g \geq 0$, prove the reverse triangle inequality $\|f+g\|_p \geq \|f\|_p + \|g\|_p$.

Exercise 1.2
Show Proposition 1.1.3 (on simple properties of d_f) in the notes.

Exercise 1.3
Prove the marked items in Proposition 1.2.5 (on simple properties of f^*) in the notes.

Exercise 1.4
Let $\hat{f}(\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot\xi} f(x)\,dx$ denote the Fourier transform which is well-defined on $L^1(\mathbb{R}^d)$ or the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and then extended to $L^p(\mathbb{R}^d)$ for any $1 \leq p \leq 2$ by Plancherel (initially on $L^1 \cap L^2$ and then extended via density to L^2) and interpolation. Recall that the interpolation lead us to the (non-optimal) Hausdorff–Young inequality
$$\|\hat{f}\|_{L^{p'}(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}\,.$$
Suppose there was an inequality of the form
$$\|\hat{f}\|_{L^q(\mathbb{R}^d)} \leq C_{p,q,d}\|f\|_{L^p(\mathbb{R}^d)}$$
for some $1 \leq p, q \leq \infty$. Show (by a scaling argument) that necessarily $q = p'$ and, by randomizing a sequence of functions and Khintchine's inequality, that $p \leq 2$.

Exercise 1.5
Let f, g be measurable on a σ-finite measure space (X,μ). Prove the Hardy–Littlewood inequality
$$\int_X |f(x)g(x)|\,d\mu(x) \leq \int_0^\infty f^*(t)g^*(t)\,dt,$$
where f^*, g^* are the decreasing rearrangements of f and g, respectively.