Harmonic Analysis
Addendum to Homework Sheet 3

Exercise 3.4
Prove Hadamard's three circle theorem. Let $g(z)$ be holomorphic on the annulus $A := \{z \in \mathbb{C} : r_1 < |z| < r_3\}$ for some $0 < r_1 < r_3$ and denote

$$M(r) := \max_{\theta \in [0,2\pi]} |g(re^{i\theta})| \quad \text{for } r \in (r_1, r_3)\,. \tag{1}$$

Prove that

$$\log(\frac{r_3}{r_1}) \log M(r_2) \leq \log(\frac{r_3}{r_2}) \log M(r_1) + \log(\frac{r_2}{r_1}) \log M(r_3) \tag{2}$$

for any $r_1 < r_2 < r_3$, i.e., $\log M(r)$ is a convex function of $\log r$. (Hint: Convince yourself that there is a vertical strip which the exponential function maps onto the annulus A.)
Compute both sides of the inequality when $g(z) = cz^\lambda$ for some constants $c \in \mathbb{C}$ and $\lambda \in \mathbb{Z}$.

Addendum
In the solution to the above exercise, we have seen that the inequality in (2) becomes an equality for $g(z) = cz^\lambda$ for any $c \in \mathbb{C}$ and $\lambda \in \mathbb{Z}$. Note that it is important to have $\lambda \in \mathbb{Z}$ since in this case z^λ is holomorphic on $\mathbb{C}$ and so of course on A, too. On the other hand, z^λ is not holomorphic on A for general $\lambda \in \mathbb{R}$ (or even $\lambda \in \mathbb{C}$); consider, e.g., the paradigm $\lambda = 1/2$.
The following theorem yields a converse to the above exercise.

Theorem 0.1. *Let $0 < r_1 < r_3$, $A := \{z \in \mathbb{C} :\ r_1 < |z| < r_3\}$, and $g : A \to \mathbb{C}$ be holomorphic. Let $M(r)$, $r_1 < r < r_3$ be defined as in* (1)*. Suppose the inequality* (2) *is an equality for all $r_2 \in (r_1, r_3)$. Then, there are $c \in \mathbb{C}$ and $\lambda \in \mathbb{Z}$ such that $g(z) = cz^\lambda$.*

The basic idea to prove Theorem 0.1 is to show that all complex derivatives of $g(z)$ coincide with those of cz^λ at some point $z_0 \in A$ and then use the analyticity of g to conclude equality for all $z \in A$. The constraint $\lambda \in \mathbb{Z}$ is enforced by the assumption that g is holomorphic. An important ingredient in the proof is precise control on the function $M(r)$. To this end, we prove the following lemma.

Lemma 0.2. *Let $0 < r_1 < r_3$, $A := \{z \in \mathbb{C} :\ r_1 < |z| < r_3\}$, and $g : A \to \mathbb{C}$ be holomorphic. Let $M(r)$, $r_1 < r < r_3$ be defined as in* (1)*. Suppose the inequality* (2) *is an equality for all $r_2 \in (r_1, r_3)$. Then, there are $c > 0$ and $\lambda \in \mathbb{R}$ such that*

$$M(r) = cr^\lambda. \tag{3}$$

Proof. Recall that if a convex function $f : I \to \mathbb{R}$ for some interval $I \subseteq \mathbb{R}$ obeys the equality

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y), \quad 0 \leq t \leq 1,\ x, y \in \mathbb{R},$$

then f must be affine linear, i.e., of the form $f(x) = \lambda x + B$ for some $\lambda, B \in \mathbb{R}$.
Note that (2) asserts that $M(r)$ is log-convex, i.e.,

$$\log(M(r_1^t r_3^{1-t})) \leq t \log(M(r_1)) + (1-t) \log(M(r_3)). \tag{4}$$

Indeed, this follows from (2) by introducing $t = \frac{\log(r_3/r_2)}{\log(r_3/r_1)}$ and noting

$$r_1^t r_3^{1-t} = \exp(t\log(r_1) + (1-t)\log(r_3)) = r_2. \tag{5}$$

The log-convexity of $M(r)$ as described in (4) and (5) is equivalent to saying that $\log(M(\mathrm{e}^r))$ is convex in r. Therefore, if (2) is an equality for all r_2, we see that $\log(M(\mathrm{e}^r)) = \lambda r + B$, i.e., $M(\mathrm{e}^r) = \mathrm{e}^B \cdot \mathrm{e}^{\lambda r}$. This is equivalent to

$$M(r) = \mathrm{e}^B \cdot \mathrm{e}^{\lambda \log(r)} = \mathrm{e}^B \cdot r^\lambda, \tag{6}$$

which concludes the proof. □

Proof of Theorem 0.1. First, (2) is equivalent to

$$\log(r_2) \cdot \log\left(\frac{M_3}{M_1}\right) + (\log(r_3)\log(M_1) - \log(r_1)\log(M_3)) = \log\left(\frac{r_3}{r_1}\right)\log(M_2). \tag{7}$$

Let us choose $\lambda, c' \in \mathbb{R}$ such that

$$\lambda = \frac{\log\left(\frac{M_3}{M_1}\right)}{\log\left(\frac{r_3}{r_1}\right)} \quad \text{and} \quad c' \cdot \log\left(\frac{r_3}{r_1}\right) = \log(r_3)\log(M_1) - \log(r_1)\log(M_3). \tag{8}$$

Then, for all $r \in (r_1, r_3)$, we have the implication

$$\lambda \log(r) + c' = \log(M(r)) \Rightarrow M(r) = cr^\lambda \quad \text{with } c = \mathrm{e}^{c'}. \tag{9}$$

Next, for every radius $r_0 \in (r_1, r_3)$ there is $z_0 \in A$ such that $|z_0| = r_0$ and

$$M(r_0) = |g(z_0)|. \tag{10}$$

Since g is continuous on the circle with radius r_0, we have

$$\frac{d}{d\theta}|g(z)|\Big|_{z=z_0} = 0, \quad z = |z_0|\mathrm{e}^{i\theta} \tag{11}$$

since $|g(z_0)|$ is the maximal value of $g(z)$ for $|z| = |z_0|$. Next, since $|g(z)| \leq M(|z|)$ for all $z \in A$, we see that $M(|z|) - |g(z)| \geq 0$. Moreover, the equality $M(|z|) - |g(z)| = 0$ holds if $z = z_0$. Thus, $M(|z|) - |g(z)|$ has a local minimum at $z = z_0$. Hence,

$$\frac{d}{dr}|g(z)|\Big|_{|z|=|z_0|} = M'(r)\Big|_{r=|z_0|}, \quad z = r\mathrm{e}^{i\theta}. \tag{12}$$

We now apply (11) and (12) to compute $\frac{d}{dz}g(z)$. To this end, for $g(z) = R(z)\mathrm{e}^{i\varphi(z)}$ we note

$$\begin{aligned} \frac{dg}{dz} &= \mathrm{e}^{i\varphi(z)}\left(\frac{dR}{dz} + iR(z)\frac{d\varphi(z)}{dz}\right) \\ &= g(z)\left(\frac{d\ln(R(z))}{dz} + i\frac{d\varphi(z)}{dz}\right) \\ &= R(z)\mathrm{e}^{i\varphi(z)}\left(\frac{1}{R(z)}\frac{dR(z)}{dz} + i\frac{d\varphi(z)}{dz}\right). \end{aligned} \tag{13}$$

Next recall, for $f : \mathbb{C} \to \mathbb{C}$ holomorphic at $z \in \mathbb{C}$ the complex derivative in polar coordinates

$$\frac{d}{dz}f(z) = \frac{\mathrm{e}^{-i\theta}}{2}\left(\frac{\partial f}{\partial r} - \frac{i}{r}\frac{\partial f}{\partial \theta}\right). \tag{14}$$

Thus, by (11) and (12),

$$\frac{dR}{dz} = \frac{\mathrm{e}^{-i\theta}}{2}|M'(r)|. \tag{15}$$

Next, recall the Cauchy–Riemann conditions for $f = u + iv$ in polar coordinates,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}. \tag{16}$$

In particular, for $\ln g(z) = \ln R + i\varphi$ we have (away from the branch cut of the complex logarithm),

$$\frac{\partial \ln(R(z))}{\partial r} = \frac{1}{r}\frac{\partial \varphi(z)}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial \ln(R(z))}{\partial \theta} = -\frac{\partial \varphi(z)}{\partial r}. \tag{17}$$

Thus, since $\partial_\theta R = 0$, we obtain

$$\frac{\partial \varphi(z)}{\partial r} = 0 \quad \text{and} \quad \frac{\partial \varphi(z)}{\partial \theta} = \frac{r\, M'(r)}{R(z)}. \tag{18}$$

Thus,

$$\frac{d\varphi(z)}{dz} = -\frac{i}{2}\frac{\mathrm{e}^{-i\theta}M'(r)}{R(r)}. \tag{19}$$

Combining this with (15), we obtain (recall (13))

$$\begin{aligned}\frac{dg}{dz} &= R(z)\mathrm{e}^{i\varphi(z)}\left(\frac{\mathrm{e}^{-i\theta}}{2R(z)}M'(r) + i\cdot\left(-\frac{i}{2}\frac{\mathrm{e}^{-i\theta}M'(r)}{R(z)}\right)\right)\\ &= R(z)\mathrm{e}^{i\varphi(z)}\cdot\frac{\mathrm{e}^{-i\theta}M'(r)}{R(z)}\\ &= M'(r)\mathrm{e}^{-i\theta}\mathrm{e}^{i\varphi(z)}.\end{aligned} \tag{20}$$

Put differently, we see

$$\frac{d}{dz}g(z)\bigg|_{z=z_0} = \psi\cdot\frac{z}{|z|}M'(|z|) = c\psi\lambda z^{\lambda-1} \tag{21}$$

for some $c \in \mathbb{R}$ and $\psi \in \mathbb{C}$ obeying $|\psi| = 1$; here we used Lemma 0.2 in the last equality.
By iterating the above arguments, we can see that the constraint $|g(z_0)| \leq M(|z_0|)$ is in fact strong enough to control also all the higher derivatives, i.e., $g(z)$ has all the same derivatives as $f(z) := \alpha z^\lambda$ at $z = z_0$ with $\alpha := c\psi$. Indeed, suppose that the derivatives of g and f do not coincide. Then we can zoom in close enough that the lowest different derivative is the dominant factor in the difference $g - f$, i.e., $g(z+z_0) \approx f(z+z_0) + az^n + \mathcal{O}(z^{n+1})$ for some $n \in \mathbb{N}$ and $a \in \mathbb{C}$. We choose a direction θ such that $a\theta^n$ has the same phase as $f(z_0)$; then, taking $z = \varepsilon\theta$ for sufficiently small ε yields $|g(z+z_0)| \approx |f(z+z_0)| + |a||z|^n + \mathcal{O}(|z|^{n+1})$, which violates the bound $|g(z+z_0)| \leq |f(z+z_0)|$ which comes from the constraint $|g(z_0)| \leq M(|z_0|)$.
Thus, having shown that $g(z)$ has all the same derivatives as αz^λ at $z = z_0$, the analyticity of $g(z)$ implies

$$g(z) = \alpha z^\lambda, \quad \text{for all } z \in A. \tag{22}$$

The claim that $\lambda \in \mathbb{Z}$ follows from the condition that g is holomorphic. $\square$