The assumption in the second statement holds, e.g., for $P(\xi) = \xi^2 - 1$. The curvature of the sphere $r\mathbb{S}^{d-1}$ goes like r^{-2}, whereas $(\nabla P)(|\xi| = r) = r$.

Proof. (1) By Bak–Seeger [2, Theorem 1.1], it suffices to show $|(d\sigma_{S_t})^\vee(x)| \lesssim_\tau < x >^{-(d-1)/2}$. Since this bound is known at $t = 0$, we are left to show

$$|(d\sigma_{S_t})^\vee(x) - (d\sigma_S)^\vee(x)| \lesssim_\tau < x >^{-(d-1)/2} \ .$$

We recall (6.7), i.e.,

$$(d\sigma_S)^\vee(x) - (d\sigma_{S_t})^\vee(x) = \int_S d\sigma_S(\xi) \mathrm{e}^{2\pi i x\cdot\xi} \Psi_{t,x}(\xi) \,, \tag{6.2}$$

where

$$\Psi_{t,x}(\xi) := \left[1 - \mathrm{e}^{2\pi i x\cdot(\psi(t)\xi - \xi)} \exp\left(\int_0^t \operatorname{div} j(\psi(\mu)\xi)\, d\mu\right)\right] .$$

Decomposing $\Psi_{t,x}(\xi)$ on S smoothly into (sufficiently small) compactly supported functions, say $\{\Psi_{t,x}(\xi)\chi_j(\xi)\}_{j=1}^N$ for $\chi_j \in \mathbb{C}_c^\infty(\Omega)$ and some $N \in \mathbb{N}$, shows that there is, for every $x \in \mathbb{R}^d \setminus \{0\}$, at most one point $\bar\xi = \bar\xi(x) \in S$ with a normal pointing in the direction of x. Then, by stationary phase arguments, Hlawka [10] and Herz [9] (see also Stein [19, p. 360]) already showed that the leading order in the asymptotic expansion (as $|x| \to \infty$) of (6.2) with the cut off amplitude $\Psi_{t,x}\chi_j$ is given by

$$|x|^{-(d-1)/2} \Psi_{t,x}(\bar\xi(x))\chi_j(\bar\xi(x)) |K(\bar\xi(x))|^{-1/2} \mathrm{e}^{-i\pi d/4} .$$

Here, $|K(\xi)|$ is the absolute value of the Gaussian curvature at $\bar\xi(x)$. But since $|\Psi_{t,x}(\xi)| \lesssim_\tau 1$ on S, we are done.

(2) Recall that our restriction and extension operators F_{S_t} and $F_{S_t}^*$ are defined with respect to the canonical measure $d\sigma_{S_t}(\xi) = |\nabla P(\xi)|^{-1} d\Sigma_{S_t}(\xi)$, where $d\Sigma_{S_t}$ is the induced (Lebesgue) surface measure. Since the Fourier transform $\widehat{d\Sigma_{S_t}}(x)$ is to leading order in $|x|$ proportional to $|K(\bar\xi(x))|^{-1/2} < x >^{-(d-1)/2}$ (where $\bar\xi(x)$ is some point in a patch of S_t where the cut-off measure $d\Sigma$ is Fourier transformed), see, e.g., Stein [19, p. 360]), the claim follows. □

We are now ready to prove the Hölder continuity for more general dispersion surfaces.

Lemma 6.2. *Assume $T(\xi)$ satisfies the assumptions stated at the beginning of Section 2. Let $1 \leq p < p_c$, $1/q = 1/p - 1/p'$, i.e., $1 \leq q < (d+1)/2$, and $0 < \alpha < \min\{(d+1)/2 - q, 1\}$. Then, the assertions of Lemma 4.1 hold.*

In fact, $P \in C^{\lceil d/2 \rceil + 4}(\Omega)$ would suffice for our purposes, but see also Remarks 2.4.

Proof. First note that $S_t = S_t^o \cup S_t^i$ where S_t^o, S_t^i lie outside, respectively inside S. Thus, $F_{S_t}^* F_{S_t} - 2F_S^* F_S = (F_{S_t^o}^* F_{S_t^o}) - F_S^* F_S + (F_{S_t^i}^* F_{S_t^i}) - F_S^* F_S$ and it suffices to consider only one of these surfaces. In the following, we treat the outer one and abuse notation by writing $S_t \equiv S_t^o$. By Lemma 4.1, it suffices to prove

$$|(d\sigma_{S_t})^\vee(x) - (d\sigma_S)^\vee(x)| \lesssim t^\alpha (1 + |x|)^{\alpha - (d-1)/2} \tag{6.3}$$

for all $t \in (0, \tau)$ and $\alpha = 0, 1$. But since

$$|(d\sigma_S)^\vee(x)| \lesssim (1 + |x|)^{-(d-1)/2}$$

by the assumptions on T and stationary phase, and $t \in [0, \tau]$ for some fixed τ, it suffices to prove (eq:ptboundmugen 6.3) for $\alpha = 1$.

We will now express $(d\sigma_{S_t})^\vee$ in terms of $(d\sigma)^\vee$. By standard facts from differential geometry (see, e.g., Yafaev [Yafaev2010 23, Chapter 2, Section 1]), we have

$$\int_{S_t} \mathrm{e}^{2\pi i x \cdot \xi} d\sigma_{S_t}(\xi) = \int_S \mathrm{e}^{2\pi i x \cdot \psi(t)\xi} \tau(t, \xi)\, d\sigma_S(\xi)\,. \tag{6.4}$$ eq:ftmeasuret

Here $\psi(t) : S \to S_t$ is a diffeomorphism defined by the formula

$$\psi(t)\zeta = \xi(t)\,, \quad \zeta \in S \tag{6.5}$$ eq:defdiffeo

where $\xi(t)$ solves the differential equation

$$\begin{cases} \frac{d\xi(t)}{dt} = j(\xi(t)) \\ \xi(0) = \zeta \in S \end{cases}$$

and

$$j(\xi) = \frac{\nabla P(\xi)}{|\nabla P(\xi)|^2} \in C^\infty(P^{-1}[0, t])\,,$$

i.e., $j(\xi(t))$ is a flow along the normals of S_t. Moreover,

$$\tau(t, \xi) = \frac{d\sigma_{S_t}(\psi(t)\xi)}{d\sigma_S(\xi)}\,, \quad \xi \in S \tag{6.6}$$ eq:rnderivative

is the Radon–Nikodým derivative of the preimage of the measure $d\sigma_{S_t}$ under the mapping $\psi(t)$ with respect to the measure $d\sigma_S$. By [Yafaev2010 23, Chapter 2, Lemma 1.9] it can be expressed as

$$\tau(t, \xi) = \exp\left(\int_0^t (\operatorname{div} j)(\psi(\mu)\xi)\, d\mu \right)\,, \quad \xi \in S\,.$$

Thus, we have

$$(d\sigma_S)^\vee(x) - (d\sigma_{S_t})^\vee(x) = \int_S d\sigma_S(\xi) \mathrm{e}^{2\pi i x \cdot \xi} \left[1 - \mathrm{e}^{2\pi i x \cdot (\psi(t)\xi - \xi)} \exp\left(\int_0^t \operatorname{div} j(\psi(\mu)\xi)\, d\mu \right) \right]. \tag{6.7}$$ eq:difftmeas

By the mean value theorem, there is a $\tilde{t} \in [0, t]$ such that

$$\begin{aligned} & 1 - \mathrm{e}^{2\pi i x \cdot (\psi(t)\xi - \xi)} \exp\left(\int_0^t \operatorname{div} j(\psi(\mu)\xi)\, d\mu \right) \\ & = t \left[x \cdot \frac{d\psi(t)\xi}{dt}(\tilde{t}) + \operatorname{div} j(\psi(\tilde{t})\xi) \right] \cdot \mathrm{e}^{2\pi i x \cdot (\psi(\tilde{t})\xi - \xi)} \exp\left(\int_0^{\tilde{t}} \operatorname{div} j(\psi(\mu)\xi)\, d\mu \right) \\ & = t \left[x \cdot j(\xi(\tilde{t})) + \operatorname{div} j(\xi(\tilde{t})) \right] \cdot \mathrm{e}^{2\pi i x \cdot (\xi(\tilde{t}) - \xi)} \exp\left(\int_0^{\tilde{t}} \operatorname{div} j(\xi(\mu))\, d\mu \right)\,. \end{aligned}$$

Writing $x = |x|\eta$ with $\eta \in \mathbb{S}^{d-1}$, we are left to show

$$\left| \int_S \left[j(\xi(\tilde{t})) \cdot \eta + \operatorname{div} j(\xi(\tilde{t})) \right] \exp\left(\int_0^{\tilde{t}} \operatorname{div} j(\xi(\mu))\, d\mu \right) \cdot \mathrm{e}^{2\pi i x \cdot \xi(\tilde{t})}\, d\sigma_S(\xi) \right| \lesssim (1+|x|)^{-\frac{d-1}{2}}\,, \tag{6.8}$$

eq:perturbation

uniformly in $\tilde{t} \in [0,t]$ and $t \in (0,\tau)$. If τ is chosen small enough, then the integrand can be understood as a tiny perturbation of the integrand when $\tilde{t} = 0$. In this situation, the assertion is well known and its proof can be found, e.g., in Stein [19, Chapter VIII, Theorem 1]. Let us first rewrite the left side of (6.8) as

$$\left| \int_S F(\xi(\tilde{t}), \xi(\mu)) \cdot \mathrm{e}^{2\pi i x \cdot \xi(\tilde{t})}\, d\sigma_S(\xi) \right|$$

with

$$F(\xi(\tilde{t}), \xi(\mu)) = \left[j(\xi(\tilde{t})) \cdot \eta + \operatorname{div} j(\xi(\tilde{t})) \right] \exp\left(\int_0^{\tilde{t}} \operatorname{div} j(\xi(\mu))\, d\mu \right) \chi_\tau(\xi)\,,$$

where χ_τ was defined after (3.7). Since $F \in C_c^\infty(\mathbb{R}^d)$ [2] whose support intersects S in a compact subset of S, we can apply a stationary phase argument and repeat the strategy in [19, Chapter VIII, Theorem 1]. To this end we describe S, locally at least, by the graph of a C^∞ function. More precisely, let ξ_0 be any point of S and consider a rotation and translation of the ambient $\mathbb{R}^d$ such that ξ_0 is moved to the origin, and the tangent plane to S at ξ_0 becomes the hyperplane $\xi_d = 0$. Then near the origin (i.e., near ξ_0), the surface S can be given as a graph $\xi_d = \varphi(\xi')$ (with $\xi' = (\xi_1, ..., \xi_{d-1})$) for some $\varphi \in C^\infty(\mathbb{R}^{d-1})$ with $\varphi(0) = \nabla\varphi(0) = 0$. Moreover, since S is supposed to have non-vanishing Gaussian curvature, we have

$$\det\left(\frac{\partial^2 \varphi}{\partial \xi_j \partial \xi_k} \right)(\xi_0) \neq 0\,, \tag{6.9}$$

eq:nondegcond

i.e., the principal curvatures do not vanish at $\xi_0 = 0$. Thus, $d\sigma_S(\xi) = (1+|\nabla\varphi|^2)^{1/2}\, d\xi'$ and we are left to consider

$$\left| \int_{\mathbb{R}^{d-1}} G(\xi(\tilde{t}), \xi(\mu)) \cdot \mathrm{e}^{2\pi i |x| \Phi(\eta, \xi(\tilde{t})')}\, d\xi' \right|$$

with $\xi(\tilde{t}) = \psi(\tilde{t})(\xi', \varphi(\xi'))$ and $G(\xi(\tilde{t}), \xi(\mu)) := F(\xi(\tilde{t}), \xi(\mu))(1+|\nabla\varphi(\xi')|^2)^{1/2}$. Moreover,

$$\begin{aligned} \Phi(\eta, \xi(\tilde{t})) &:= \eta \cdot \psi(\tilde{t})(\xi', \varphi(\xi')) = \eta \cdot \left[(\xi', \varphi(\xi')) + \tilde{t} j(\psi(\tilde{t}_1)\xi) \right] \\ &= \sum_{j=1}^{d-1} \eta_j \xi_j + \eta_d \varphi(\xi') + \tilde{t}\eta \cdot j(\psi(\tilde{t}_1)\xi) \end{aligned}$$

[2]In fact, if P is assumed to be in $C^{4+\lceil d/2 \rceil}(\Omega)$, then $F \in C_c^{2+\lceil d/2 \rceil}(\Omega)$ which just suffices to run the classic stationary phase argument to obtain the estimate $|(d\mu)^\vee(x)| \lesssim |x|^{-(d-1)/2}$.

where we used the mean value theorem for some $\tilde{t}_1 \in (0, \tilde{t})$. We distinguish now between three cases, depending on the position of $\eta \in \mathbb{S}^{d-1}$, namely

(1) η is sufficiently close to the "north pole" $\eta_N = (0, 0, ..., 1)$,
(2) η is sufficiently close to the "south pole" $\eta_S = (0, 0, ..., -1)$, and
(3) η lies in the complementary set on the unit sphere.

By the arguments of the proof of [19, Chapter VIII, Theorem 1], the proof is concluded, once we show that, for sufficiently small τ,

$$\det\left(\frac{\partial^2 \Phi}{\partial \xi_j \partial \xi_k}\right)(\eta_N, \xi' = 0) \neq 0 \tag{6.10a}$$

(and analogously for η_S instead of η_N) and

$$|\nabla_{\xi'} \Phi(\eta, \xi(\tilde{t}))| > 0 \tag{6.10b}$$

for η lying in the complementary set on the unit sphere. We start with the verification of (6.10b) and compute

$$\nabla_{\xi'} \Phi(\eta, \xi(\tilde{t})) = \eta' + \eta_d \nabla_{\xi'} \varphi(\xi') + \tilde{t} \nabla_{\xi'} \eta \cdot j(\xi(\tilde{t}_1))\,.$$

Since $\nabla \varphi(\xi') = \mathcal{O}(\xi')$ as $\xi' \to 0$ and $j \in C^1(P^{-1}[0,t])$, we see that $|\nabla_{\xi'} \Phi(\eta, \xi(\tilde{t}))| > 0$ if τ is chosen sufficiently small and the support of G is a sufficiently small neighborhood of the origin. Let us now finally verify (6.10a). Since

$$\frac{\partial^2 \Phi(\eta, \xi(\tilde{t}))}{\partial \xi_j \partial \xi_k} = \eta_d \frac{\partial^2 \varphi(\xi')}{\partial \xi_j \partial \xi_k} + \tilde{t} \eta \cdot \frac{\partial^2 j(\xi(\tilde{t}_1))}{\partial \xi_j \partial \xi_k}$$

the non-degeneracy condition (6.9) implies that also (6.10a) holds, if t and the support of G are sufficiently small. This concludes the proof of Lemma 6.2. □

7. A larger class of admissible potentials

Let us now see how to incorporate more general potentials. First, we show how to generalize Theorem 2.5 on weighted L^2 spaces to Agmon–Hörmander spaces.

Afterwards we show how to get rid of some local regularity assumptions.

7.1. Generalization to Agmon–Hörmander spaces.

7.2. Weaker local regularity assumptions.

Let $\{Q_s\}_{s \in \mathbb{Z}^d}$ be a collection of axis-parallel cubes such that $\mathbb{R}^d = \cup_s Q_s$. For $1 \leq q_1, q_2 < \infty$ we introduce the norms

$$\|V\|_{\ell^{q_2} L^{q_1}} := \Big[\sum_s \|V\|_{L^{q_1}(Q_s)}^{q_2}\Big]^{1/q_2},$$
$$\|V\|_{\ell^\infty L^{q_1}} := \sup_s \|V\|_{L^{q_1}(Q_s)}.$$

Lemma 7.1. *Let $\lambda \geq 1$, $\eta \in C_c^\infty(\mathbb{R}^n)$ a bump function adapted to $1/2 \leq |\xi| \leq 2$ and $2r' \geq p_n'$ (which means $1 \leq r \leq (n+1)/2$). Then*

$$\||V|^{1/2} \eta(p/\lambda)\|_{p_n' \to 2} + \|\eta(p/\lambda)|V|^{1/2}\|_{2 \to p_n} \lesssim \lambda^{n(\frac{1}{p_n'} - \frac{1}{2r'})} \|V\|_{\ell^{\frac{n+1}{2}} L^r}^{1/2}. \tag{7.1}$$