

6 Connection between restriction and Kahya's problem

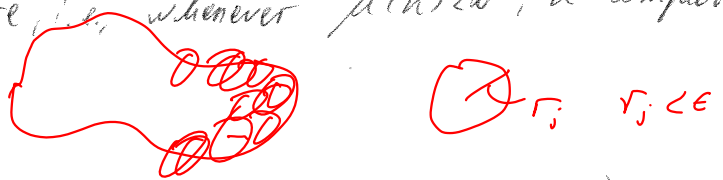
(Mattila - Fourier Analysis & Hausdorff Dimension 2015 - Geometry of sets & measures sect 4.3
 Wolff - Lectures on Harmonic Analysis 2003)
 Federer - Sect 2.10
 Falconer - Ch I, III set

6.1 ~~Hausdorff~~ measures

6.1 ~~restriction~~ → Kahya maximal conjecture

6.1 Hausdorff measures

- By "measure" on a set X we usually mean an outer measure, i.e., a function $\mu: \{A: A \subseteq X\} \rightarrow \mathbb{R}_{\geq 0}$ with $\mu(\emptyset) = 0$ which is monotone and countably subadditive, i.e., $\mu(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$ for any countable, disjoint family $\{E_n\}_{n=1}^{\infty}$ on which μ is defined
- Borel sets in a metric space X form smallest σ -algebra of subsets of X containing all open (or closed) subsets of X
- Borel measure in X is a measure μ for which Borel sets are measurable and which is regular in sense that $\forall A \subseteq X \exists$ Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$
- Radon measure := locally finite Borel measure, i.e., whenever $\mu(K) < \infty$, K compact
- restriction



Spherical Hausdorff measures

Fix $\alpha > 0$, and let $E \subseteq \mathbb{R}^d$. For $\epsilon > 0$, define $H_{\alpha}^{\epsilon}(E) := \inf \left(\sum_{j=1}^{\infty} r_j^{\alpha} \right)$ where infimum is taken over all countable coverings of E by discs $B_{x_j}(r_j)$ with $r_j < \epsilon$

Clearly: $H_{\alpha}^{\epsilon}(E)$ increases as ϵ decreases

Define $H_{\alpha}(E) := \lim_{\epsilon \rightarrow 0} H_{\alpha}^{\epsilon}(E)$

Clearly: $H_{\alpha}^{\epsilon}(E) \leq H_{\beta}^{\epsilon}(E)$ if $\alpha > \beta$ and $\epsilon \leq 1$

$\Rightarrow H_{\alpha}(E)$ is non-increasing in α

H_1, H_0

Remarks 6.1 1) If $H_{\alpha}^{\delta}(E) = 0$, then $H_{\alpha}(E) = 0$. (For, a covering with $H_{\alpha}^{\delta}(E) < \delta$ necessarily consists of balls with radii $< \delta^{1/\alpha}$ by pigeonhole principle)

2) $H_{\alpha}(E) = 0$ whenever $\alpha > d$ (see Lemma below + next remark)

3) $H_d(E) = \frac{1}{\text{Leb}(B_0(1))} \text{Leb}(E)$ (Federer Thm 2.10.35 with notation given in 2.10.2)

Lemma 6.2 Let $E \subseteq \mathbb{R}^d$, then there is a unique number α_0 , called Hausdorff dimension of E , $\dim_H E$ s.t. $\forall \alpha \in \mathbb{R}$ $H_\alpha(E) = \infty$ if $\alpha < \alpha_0$ and $H_\alpha(E) = 0$ if $\alpha > \alpha_0$ (2)

Prf Define $\alpha_0 := \sup \{ \alpha > 0 : H_\alpha(E) = \infty \} \rightarrow H_\alpha(E) = 0 \forall \alpha < \alpha_0$ (since $\alpha \mapsto H_\alpha(E)$ nonincreasing)

For $\alpha > \alpha_0$ let $\beta \in (\alpha_0, \alpha)$ and $M := 1 + H_\beta(E) < \infty \rightarrow$ for $\epsilon > 0$ we have a covering

$$\bigcup B_{r_j} \supseteq E, \quad r_j < \epsilon \quad \text{with} \quad \sum r_j^\beta \leq M$$

$$\Rightarrow \sum r_j^{\alpha - \beta + \beta} \leq \epsilon^{\alpha - \beta} \sum r_j^\beta \leq M \epsilon^{\alpha - \beta} \xrightarrow{\epsilon \rightarrow 0} 0 \Rightarrow H_\alpha(E) = 0 \forall \alpha > \alpha_0 \quad \square$$

Remarks 6.3 1) Lemma 6.2 \Leftrightarrow for $0 \leq \beta < \alpha$ and $E \subseteq \mathbb{R}^d$ we have

$$H_\beta(E) < \infty \Rightarrow H_\alpha(E) = 0$$

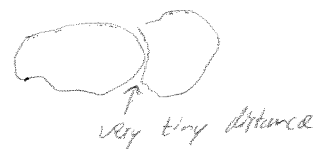
$$H_\alpha(E) > 0 \Rightarrow H_\beta(E) = \infty$$

2) H_α is countably additive on Borel sets $H_\alpha(\bigcup E_j) = \sum H_\alpha(E_j)$ and in particular H_α defines a Borel measure

Note that σ -additivity generally fails for H_α^e : ~~example H_α^e~~

example $H_\alpha(E) + H_\alpha(F) = H_\alpha(E \cup F)$, but

$$H_\alpha^e(E \cup F) \neq H_\alpha^e(E) + H_\alpha^e(F) \quad \text{for compact disjoint } E, F \subseteq \mathbb{R}^d$$



3) H_α not σ -finite if $\alpha < d$ since already $H_\alpha(E) = \infty$ for all $E \subseteq \mathbb{R}^d$ with $\text{Leb}(E) > 0$

4) Standard example of a set $C \subseteq \mathbb{R}$ with $\dim C \in (0, 1)$ is Cantor's set $\dim C_\gamma = \frac{\log 2}{\log 1/\gamma}$, $\gamma \in (0, \frac{1}{2})$ (start with $[0, 1]$ and remove middle segment of length $1-2\gamma$) (choice: $\gamma = \frac{1}{3}$)

Although estimating $\dim E$ from above is easy (by just estimating convenient coverings by variational principle), often lower bounds are much harder to obtain as all coverings must be evaluated.

The following lemma due to Frostman (cf. Wolff Prop 8.2 or Mattila 2015, Thm 2.7) transforms the problem of finding measures with good upper bounds to for measures with balls

We state Frostman's lemma only for compact sets. For Borel or more general (Suslin or analytic) sets, see eg Bishop and Peres (2016) or Carleson (1967)

If $E \subseteq \mathbb{R}^d$ is compact, we denote by $\mathcal{P}(E)$ the set of all probability measures supported on E .

Proposition 6.4 (Frostman's lemma) Let $0 \leq \alpha \leq d$ and $E \subseteq \mathbb{R}^d$ compact.

\Rightarrow Then: $H_\alpha(E) > 0 \Leftrightarrow \exists \mu \in \mathcal{P}(E)$ s.t. $\mu(B_x(r)) \leq cr^\alpha$ for all $x \in \mathbb{R}^d$, $r > 0$ and a suitable const $c > 0$

(Equivalently: $\dim(E) = \sup\{0 \leq \alpha \leq d : \exists \mu \in \mathcal{P}(E) : \mu(B_x(r)) \leq cr^\alpha \forall x \in \mathbb{R}^d, r > 0\}$)

"Proof" " \Leftarrow " Let $\{B_{x_i}(r_i)\}$ be any covering of E , then

$$1 = \mu(E) \leq \sum \mu(B_{x_i}(r_i)) \leq c \sum r_i^\alpha$$

which shows $H_\alpha(E) \geq \frac{1}{c}$.

" \Rightarrow " The construction of a suitable measure is most easily done using dyadic cubes.

$$\mathcal{Q}_k = \{2^{-k} \cdot n + [0, 2^{-k}]^d\}_{n \in \mathbb{Z}^d} \quad \text{mesh of dyadic cubes of side length } 2^{-k} \text{ with vertices at } 2^{-k}\mathbb{Z}^d$$

Recall the nesting properties: if $Q \in \mathcal{Q}_k$, then there is a unique $\tilde{Q} \in \mathcal{Q}_{k-1}$

such that $Q \subseteq \tilde{Q}$. Furthermore, if we fix $Q_1 \in \mathcal{Q}_{k-1}$, then Q_1 is the union of those $Q \in \mathcal{Q}_k$ with $\tilde{Q} = Q_1$ and the union is disjoint except for edges and vertices.

For a dyadic cube Q , there is $B_x(r) \supseteq Q$ and $r \leq c \cdot \ell(Q)$ (length of edge of Q)

Likewise, if one fixes $B_x(r)$, then there is a bounded number of dyadic cubes Q_1, \dots, Q_c with $\ell(Q_j) \leq Cr$ and $\bigcup_{j=1}^c Q_j \supseteq B_x(r)$

\Rightarrow definition of Hausdorff measure and also the statement " $\mu(B_x(r)) \leq cr^\alpha$ " in this proposition, could equally well be given in terms of dyadic cubes.

\Rightarrow except for the values of involved constants:

$$\mu(B_x(r)) \leq cr^\alpha \quad \forall x \in \mathbb{R}^d, r > 0, \quad (\Leftrightarrow) \quad \mu(Q) \leq c \ell(Q)^\alpha \quad \text{for all dyadic cubes}$$

suitable c

Furthermore, if we define

$$h_\alpha^c(E) := \inf \left(\sum_{Q \in \mathcal{F}} \ell(Q)^\alpha : E \subseteq \bigcup_{Q \in \mathcal{F}} Q \right)$$

where \mathcal{F} runs over all coverings of E by dyadic cubes of side length $\ell(Q) < \epsilon$

and $h_\alpha(E) := \lim_{\epsilon \rightarrow 0} h_\alpha^c(E)$,

then we have $C^{-1} h_\alpha^\epsilon(E) \leq h_\alpha^\epsilon(E) \leq C h_\alpha^\epsilon(E)$ and therefore (4)

$$h_\alpha(E) > 0 \Leftrightarrow H_\alpha(E) > 0$$

Now we return to the actual proof of Frostman.

Wlog assume $E \subseteq [0, 1]^d$ and $h_\alpha'(E) > 0$ (recall Remark 6.1, 1))

Task Find $\mu \in \mathcal{P}(E)$ s.t. $\mu(Q) \leq C l(Q)^\alpha$ for all dyadic cubes with $l(Q) \leq 1$

We make a further reduction:

Claim For each fixed $m \in \mathbb{N}$ it suffices to find a positive measure μ satisfying

(1) μ is supported on the union of cubes $Q \in \mathcal{Q}_m$ which intersect E

(2) $\|\mu\| \geq \frac{1}{C}$ (C is independent of m !!)

(3) $\mu(Q) \leq l(Q)^\alpha$ for all dyadic cubes with $l(Q) \geq 2^{-m}$.

Suppose this can be done and call the resulting measure μ_m

By (3) we have bound on $\|\mu_m\|$, so that there is a weak- $*$ -limit μ

of $\mu_m \xrightarrow{m \rightarrow \infty} \mu$, which is, by (1), in particular supported on E . Moreover,

by (2) and (3) it suffices $\|\mu\| \geq \frac{1}{C}$ and $\mu(Q) \leq l(Q)^\alpha$ for dyadic cubes with $l(Q) \geq 0$

\Rightarrow a suitable scalar multiple of μ gives the desired measure $l(Q) \geq 0$

There are multiple ways to construct the measures μ_m satisfying (1)-(3); the issue is that (2) and (3) are competing conditions, i.e., one must find a measure μ with appropriate support and with total mass roughly as large as possible given that (2) holds.

We follow Carleson (1967), Ch. 2 to that end. (Constructive!)

Fix $m \in \mathbb{N}$ and construct a finite sequence of measures $\nu_m, \nu_{m-1}, \dots, \nu_0$ in that order; ν_0 will then be the measure we want.

We start by defining ν_m to be the unique measure satisfying

(4) On each $Q \in \mathcal{Q}_m$, ν_m is a scalar multiple of Lebesgue measure

(5) If $Q \in \mathcal{Q}_m$ and $Q \cap E = \emptyset$, then $\nu_m(Q) = 0$

(6) If $Q \in \mathcal{Q}_m$ and $Q \cap E \neq \emptyset$, then $\nu_m(Q) = 2^{-m\alpha}$

If we set $h=m$, then ν_h has the following properties

(A) $\nu_h \ll \text{Leb}$

(B) $\nu_h(Q) \leq l(Q)^\alpha$ if $Q \in \mathcal{Q}_j$, $h \leq j \leq m$

(C) If $\alpha, \epsilon \in \mathbb{R}_{>0}$, $Q_1 \in \mathcal{Q}_h$, then there is a covering \mathcal{F}_α of $Q_1 \cap E$ by dyadic cubes in Q_1 s.t. $\nu_h(Q_i) \geq \sum_{Q \in \mathcal{F}_\alpha} l(Q)^\alpha$

Now let $1 \leq h \leq m$ and assume we have constructed an absolutely continuous \mathbb{R}^n measure ν_h with properties (1), (B) and (C)

We will now construct ν_{h-1} , having the same properties where in (B) and (C) h is replaced by $h-1$.

Namely, to define ν_{h-1} , it suffices to define $\nu_{h-1}(Y)$ when Y is contained in a cube $Q \in \mathcal{Q}_{h-1}$. Fix $Q \in \mathcal{Q}_{h-1}$, and consider two cases

(i) $\nu_h(Q) \leq \ell(Q)^\alpha$. In this case we let ν_{h-1} agree with ν_h on subsets of Q

(ii) $\nu_h(Q) > \ell(Q)^\alpha$. In this case we let ν_{h-1} agree with $c \cdot \nu_h$ on subsets of Q ,

$$\text{where } c := \frac{\ell(Q)^\alpha}{\nu_h(Q)} \quad c := \frac{2^{-(h-1)\alpha}}{\nu_h(Q)}$$

Notice that $\nu_{h-1}(Y) \leq \nu_h(Y)$ for any set Y and furthermore $\nu_{h-1}(Q) \leq \ell(Q)^\alpha$ whenever $Q \in \mathcal{Q}_{h-1}$. These properties and (B) for ν_h imply (B) for ν_{h-1} . Moreover (1) for ν_{h-1} follows trivially from (1) for ν_h . To see that ν_{h-1} also satisfies (C), fix $Q \in \mathcal{Q}_{h-1}$. If Q is as in case (ii) then $\nu_{h-1}(Q) = \ell(Q)^\alpha$, so we can use the covering by the singleton $\{Q\}$. If Q is as in case (i), then for each of the cubes $Q_j \in \mathcal{Q}_h$ whose union is Q , we have the covering of $Q_j \cap E$ associated with (B) for ν_h . But since ν_h and ν_{h-1} agree with each other on subsets of Q , we can simply put these coverings together to obtain a suitable covering of $Q \cap E$. This shows (B) for ν_{h-1} and concludes the inductive step $\nu_h \rightarrow \nu_{h-1}$.

\Rightarrow We have therefore constructed ν_0 . It has properties (1) and (3) (since for ν_0 this is equivalent to $\nu_0(E) < \infty$ (B)), and by (C) and the definition of ν_h it has property (2) \square

We will now transform Frostman's lemma ($\dim E = \sup_{0 < \alpha \leq d} \{0 \leq \alpha \leq d: \exists \mu \in \mathcal{P}(E) \text{ s.t. } \mu(B_x(r)) \leq r^\alpha\}$)

into an integral condition related to Riesz transforms ("Coulombic energies")

~~Unfortunately, the stronger version~~



$$\sum_j \delta(x - x_j)$$



$$\sum_{j \neq l} \int \frac{\delta(x - x_j) \delta(y - x_l)}{|x - y|} dx dy$$

α -dimensional (Coulombic) energy of a positive measure μ (6)

$$\rightarrow I_\alpha(\mu) := \int \frac{d\mu(x) d\mu(y)}{|x-y|^\alpha}, \quad 0 < \alpha < d$$

\leftarrow Riesz kernel

For simplicity, suppose μ is compactly supported (not needed though in the following)

In this case, $I_\alpha(\mu) < \infty \Rightarrow I_\beta(\mu) < \infty$, whenever $\alpha > \beta > 0$ (see lemma below for details)

Define the "mean-field potential"

$$V_\mu^\alpha(x) := \int \frac{d\mu(y)}{|x-y|^\alpha}, \quad \text{i.e., } I_\alpha(\mu) = \int V_\mu^\alpha(x) d\mu(x).$$

(Note that $\Delta V_\mu^\alpha = 0$ for $x \notin \text{supp}(\mu)$ since $\Delta \frac{1}{|\cdot|^{d-2}} = 0$ away from origin)

Note also $V_\mu^\alpha(x) = \alpha \int_0^\infty \frac{\mu(B_x(r))}{r^\alpha} \frac{dr}{r}$ since $\alpha \int_0^\infty \frac{dr}{r^{\alpha+1}} \int_{|x-y|<r} d\mu(y) = \alpha \int d\mu(y) \int_{|x-y|<r} \frac{dr}{r^{\alpha+1}} = \int \frac{d\mu(y)}{|x-y|^\alpha}$

Unfortunately the stronger equivalence $I_\alpha(\mu) < \infty \Leftrightarrow \exists \mu \in \mathcal{P}(E) : \mu(B(r)) \leq r^\alpha$ is false!
 (compared to $\dim E = \sup \{0 < \alpha < d : \exists \mu \in \mathcal{P}(E) : I_\alpha(\mu) < \infty\}$
 $\sup \{\alpha \in [0, d] : \exists \mu \in \mathcal{P}(E) : \mu(B(r)) \leq r^\alpha \forall r\}$ (by Frostman))

Lemma 6.5 (1) If μ is a compactly supported probability measure satisfying $\mu(B(r)) \leq r^\alpha$, then $I_\beta(\mu) < \infty \forall \beta < \alpha$

(2) If μ is a compactly supported probability measure satisfying $I_\alpha(\mu) < \infty$, then there is another probability measure ν s.t. $\nu(X) \leq 2\mu(X)$ for all sets X and such that $\nu(B(r)) \leq r^\alpha$

Proof (1) Suppose wlog $\text{diam}(\text{supp } \mu) \leq 1$, then $V_\mu^\beta(x) \leq \sum_{j \geq 0} 2^{j\beta} \mu(B_x(2^{-j}))$

Accordingly, if $\mu(B(r)) \leq r^\alpha$ and $\beta < \alpha$, then $V_\mu^\beta(x) \leq \sum_{j \geq 0} 2^{j(\beta-\alpha)} \leq_{\text{u.p.}} 1$

$$\Rightarrow \int V_\mu^\beta(x) d\mu(x) = I_\beta(\mu) \leq_{\text{u.p.}} 1$$

(2) Let $F := \{x \in \mathbb{R}^d : V_\mu^\alpha(x) \leq 2I_\alpha(\mu)\}$. Then, $\mu(F) = \int \mathbb{1}_{\{V_\mu^\alpha(x) \leq 2I_\alpha(\mu)\}} d\mu(x)$
 $= \frac{1}{2I_\alpha(\mu)} \int 2I_\alpha(\mu) \mathbb{1}_{\{V_\mu^\alpha(x) \leq 2I_\alpha(\mu)\}} d\mu(x)$
 $\leq \frac{1}{2I_\alpha(\mu)} \int V_\mu^\alpha(x) d\mu(x) = \frac{I_\alpha(\mu)}{2I_\alpha(\mu)} = \frac{1}{2}$

Let ν be defined by $\nu(X) = \frac{\mu(X \cap F)}{\mu(F)}$. Then $\nu(X) \leq 2\mu(X \cap F) \leq 2\mu(X)$, so we are left with showing $\nu(B(r)) \leq r^\alpha, \forall x \in \mathbb{R}^d$

Supp Suppose first $x \in F$. Then $v \in 2\mu$ $x \in F \Rightarrow \int V_\mu^\alpha(x) \in 2 \int V_\mu^\alpha(x)$ $\int V_\mu^\alpha(x) \in 2 \int V_\mu^\alpha(x)$ $\int V_\mu^\alpha(x) \in 2 \int V_\mu^\alpha(x)$

$$v(B_x(r)) = \int_{B_x(r)} \frac{d\nu(y)}{|x-y|^\alpha} \quad \alpha - \gamma |x|^\alpha \leq r^\alpha V_\nu^\alpha(x) \leq 2r^\alpha V_\mu^\alpha(x) \leq 4r^\alpha I_\alpha(\mu)$$

Now assume $x \notin F$ and distinguish between the following two cases.

(a) Recall $v(X) = \frac{\mu(X \cap F)}{\mu(F)}$, so if r is so small that $F \cap B_x(r) = \emptyset$, then

$v(B_x(r)) = 0$ trivially; if $B_x(r) \cap F \neq \emptyset$, let $y \in B_x(r) \cap F$. Then

$$\text{Then } v(B_x(r)) = \int d\nu(z) \frac{\mu(B_x(r) \cap F)}{\mu(B_x(r) \cap F)} \leq v(B_y(2r)) \leq r^\alpha$$

contains the above y 's; make set bigger by do incorporating balls around these y 's

$y \in B_x(r) \cap F$ $y \in F$, can use above result.

Kahane maximal \Rightarrow Kahane set

$$\int \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}$$

Proposition 6.6 Let $E \subseteq \mathbb{R}^d$ compact. Then $\dim(E) = \sup\{0 \leq \alpha \leq d : \exists \mu \in \mathcal{P}(E) \text{ s.t. } I_\alpha(\mu) < \infty\}$

Frostman: $\exists \mu \in \mathcal{P}(E)$ s.t. $\mu(B_x(r)) \leq r^\alpha$

Proof Let $s := \sup\{0 \leq \alpha \leq d : \exists \mu \in \mathcal{P}(E) \text{ s.t. } I_\alpha(\mu) < \infty\}$

If $\beta < s$, then by (2) of Lemma 6.5, E supports a measure with $\mu(B_x(r)) \leq r^\beta$

\Rightarrow by Frostman (Prop 6.4) we have $H_\beta(E) > 0$, i.e., $\beta \leq \dim E \Rightarrow s \leq \dim E$

On the other hand, if $\beta < \dim E$, then, again by Frostman, E supports a measure with $\mu(B_x(r)) \leq r^{\beta+\epsilon}$ for sufficiently small $\epsilon \Rightarrow$ by Lemma 6.5 (1), we have

$$I_\beta(\mu) < \infty, \text{ i.e., } \beta \leq s \Rightarrow \dim E \leq s$$

Observe that $I_\alpha(\mu) = \int \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} = \langle \mu, | \cdot |^{-\alpha} * \mu \rangle_{\mathcal{S}' \times \mathcal{S}} = \langle \hat{\mu}, | \cdot |^{-d+\alpha} \hat{\mu} \rangle$

Corollary 6.7 Suppose μ is a compactly supported probability measure on \mathbb{R}^d with

$$\rightarrow |\hat{\mu}(\xi)| \leq |\xi|^{-\beta} \text{ for some } 0 < \beta < d/2, \text{ or, more generally}$$

$$\text{in } L^2\text{-average } \int_{B_0(N)} |\hat{\mu}(\xi)|^2 d\xi \leq N^{d-2\beta}$$

$$2 \cdot \frac{d-1}{2} = d-1$$

$$\Rightarrow \underline{\dim(\text{supp}(\mu)) \geq 2\beta}$$

Remark Recall that FT of surface measure of curved surfaces/hypersurfaces

$$\text{obeys } \beta = \frac{d-1}{2}$$

Q $\dim(\text{supp}(E)) = \sup \{ 0 \leq \alpha \leq d : \exists \mu \in \mathcal{P}(E) : \forall I_{\alpha}(\mu) < \infty \}$ (8)
 Prop 6.6

So compute $I_{\alpha}(\mu)$ for $\alpha < 2\beta$:

$$I_{\alpha}(\mu) = \int |\hat{\mu}(\xi)|^2 |\xi|^{-d+\alpha} \left(\mathbb{1}_{|\xi| \leq 1} + \mathbb{1}_{|\xi| > 1} \right) d\xi \leq \|\mu\| + \sum_{j \geq 0} 2^{-j(d-\alpha)} \int_{2^j \leq |\xi| \leq 2^{j+1}} |\hat{\mu}(\xi)|^2 d\xi$$

$$|\hat{\mu}(\xi)| \leq \|\mu\|$$

$$\int_{\mathbb{R}^d} |\xi|^{-d+\alpha} = C d \alpha$$

$$\leq \|\mu\|^2 + \sum_{j \geq 0} 2^{-j(d-\alpha)} \cdot 2^{j(d-2\beta)} \leq 1 + \|\mu\| \text{ if } 2\beta > \alpha \quad \square$$

Natural question Does a compact set with dimension α necessarily support a measure μ that satisfies $|\hat{\mu}(\xi)| \leq_{\epsilon} (1+|\xi|)^{-\alpha/2+\epsilon}$?

No, if one insists on pointwise decay, whereas yes, if L^2 -averaged decay, bc. the calculation in the proof above is reversible

In fact there are many sets with positive dimension that do not support any measure whose FT decays at infinity

Example Consider line segment $E = [0, 1] \times \{0\} \subseteq \mathbb{R}^2$ with $\dim E = 1$; but if μ is any measure on E , then $\hat{\mu}(\xi) = \hat{\mu}(\xi_1)$ only depends on ξ_1 , so in particular it doesn't decay along ξ_2 -direction at all (unless $\hat{\mu} = 0$)

More dramatically, it's actually hard to show that a set E with given dimension supports a measure s.t. $|\hat{\mu}(\xi)| \leq_{\epsilon} (1+|\xi|)^{-\alpha/2+\epsilon}$. (\rightarrow Kaufmann see, eg. Wolff, Ch 3)

0.2 ~~Restriction~~ → Kakeya conjectures

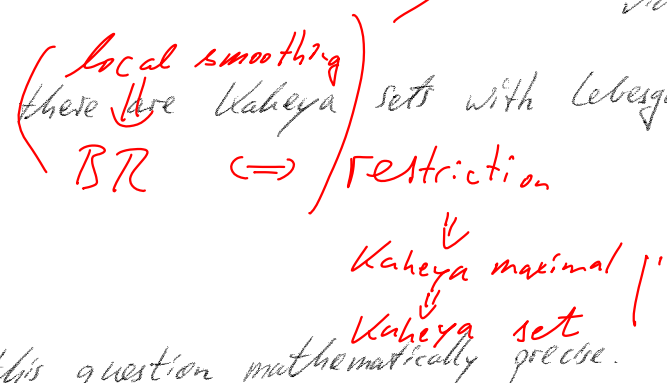
A Besicovitch, or Kakeya set is a compact set $E \subseteq \mathbb{R}^d$ which contains a unit line segment in every direction, i.e.,

$$\forall w \in S^{d-1} \exists x \in \mathbb{R}^d : x + tw \in E \quad \forall t \in [-\frac{1}{2}, \frac{1}{2}]$$



Besicovitch showed that such sets may be extremely small (\rightarrow Federer's YT video)

Thm 6.8 (Besicovitch) If $d \geq 2$, then there are Kakeya sets with Lebesgue measure zero.



But how small can they actually be?

There are several ways one can ask this question mathematically precise.

Kakeya set conjecture Besicovitch sets, necessarily have Hausdorff dimension d , in \mathbb{R}^d

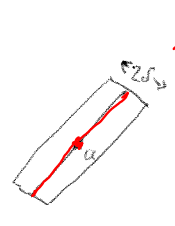
This is open in $d \geq 3$ but was initially solved by R.O. Davies 1971 in $d=2$

There is also a more quantitative formulation of this problem in terms of a maximal function

For any $0 < \delta \ll 1$ and $w \in S^{d-1}$, and $a \in \mathbb{R}^d$, introduce the δ -box around the unit line segment in direction w , centered at a as follows

$$T_w^\delta(a) := \{x \in \mathbb{R}^d : |(x-a) \cdot w| \leq \frac{1}{2}, | \underbrace{(x-a)^\perp } | \leq \delta \}$$

$$= (x-a) - \langle (x-a), w \rangle w$$



$\forall w \in S^{d-1}$

$$M_\delta f(x) = \sup_{r>0} \frac{1}{B(x,r)} \int_{B(x,r)} |f|$$

$M: L^p \rightarrow L^p$ $p \in (1, \infty]$
 $L^1 \rightarrow L^{1,\infty}$

Kakeya maximal fct for $f \in L^1_{loc}(\mathbb{R}^d)$ defined by

$$M_\delta f(x) := \int_{T_w^\delta(x)} |f|$$

Goal $\forall \epsilon > 0 \exists c_\epsilon > 0$ s.t. $\|M_\delta^* f\|_{L^p(S^{d-1})} \leq c_\epsilon \delta^{-\epsilon} \|f\|_{L^1(\mathbb{R}^d)}$, i.e., "e-losses" are admissible.

Remarks 6.9

Remarks 6.9 (1) By definition, one trivially has

(10)

$$\|f_S^*\|_\infty \leq \|f\|_\infty \quad \forall \delta$$

$$\|f_S^*\|_\infty \leq \delta^{-(d-1)} \|f\|_1$$

(2) If $d \geq 2$ and $p < \infty$, then there can be no bound of the form

$$\|f_S^*\|_q \leq c \|f\|_p \quad \text{for } \delta\text{-independent } c.$$

To that end, consider a zero-measure Kahaya set E , and let E_δ denote its δ -neighborhood, and let $f = \mathbb{1}_{E_\delta}$

$\Rightarrow f_S^*(w) = 1$ for all $w \in S^{d-1}$, so $\|f_S^*\|_q \sim 1$. But on the

other hand $\|f\|_p^p = \int_{E_\delta} dx = |E_\delta| \xrightarrow{\delta \rightarrow 0} 0$ for any $p < \infty$.

(3) The claimed $\|f_S^*\|_p \leq c \delta^{-\epsilon} \|f\|_p$ cannot hold if $p < d$.

To that end take $f = \mathbb{1}_{B_0(\delta)} \Rightarrow T_w^S(0) \supseteq B_0(\delta) \quad \forall w$, so

$$f_S^*(w) = \frac{|B_0(\delta)|}{|T_w^S(0)|} \sim \delta \Rightarrow \|f_S^*\|_p \sim \delta$$

However $\|f\|_p \sim \delta^{d/p}$ which goes faster to zero than δ , if $p < d$.

Kahaya maximal conjecture $\forall \epsilon > 0 \exists C_\epsilon > 0$ s.t. $\|f_S^*\|_{L^p(S^{d-1})} \leq C_\epsilon \delta^{-\epsilon} \|f\|_{L^p(\mathbb{R}^d)}$,

whenever $p \in [d, \infty]$.

(Clearly it suffices one proves the estimate for $p=d$, then it follows for all other p by interpolation)

This was proved in $d=2$ by Cordoba using a nice geometric argument, but is open in $d \geq 3$

This type of result is "easy" bc. Hilbert space techniques are available

Proposition 6.10 (Kahaya maximal \Rightarrow Kahaya set conjecture)

If $\|f\|_p^* \leq \epsilon S^{-\epsilon} \|f\|_p$ holds for some $p < \infty$, then Besicovitch sets in \mathbb{R}^d necessarily have Hausdorff dimension = d.

Proof Let E be a Besicovitch set and $\{B_j\}_j = \{B(x_j, r_j)\}_j$ a covering with all $r_j \leq \frac{1}{100}$

Let $J_h = \{j : 2^{-h} \leq r_j \leq 2^{-(h-1)}\}$

For every $w \in S^{d-1}$, E contains a unit line segment I_w parallel to w

Let $S_h = \{w \in S^{d-1} : |I_w \cap \bigcup_{j \in J_h} B_j| \geq \frac{1}{100h^2}\}$, then $\bigcup_{h=1}^{\infty} S_h = S^{d-1}$, since

$\sum_h |I_w \cap \bigcup_{j \in J_h} B_j| \geq \sum_h \frac{1}{100h^2} |I_w| \text{ for } |I_w| = 1$ and $\sum \frac{1}{100h^2} < 1$.

Let $f = \chi_{F_h}$, where $F_h = \bigcup_{j \in J_h} B_j(10r_j)$

For $w \in S_h$, we have $|T_w^{2^{-h}}(a_w) \cap F_h| \geq \frac{1}{100h^2} \cdot |T_w^{2^{-h}}(a_w)|$, $w \in S_h$
midpoint of I_w

$\Rightarrow \int_{T_w^{2^{-h}}(a)} f_{2^{-h}}(w) = \frac{1}{|T_w^{2^{-h}}(a)|} \int_a^{sup} f_{2^{-h}} \geq \frac{|F_h \cap T_w^{2^{-h}}(a_w)|}{|T_w^{2^{-h}}(a)|} \geq \frac{1}{100h^2}$ if $w \in S_h$

$\Rightarrow \|f_{2^{-h}}\|_p \geq \|f_{2^{-h}}\|_{L^p(S_h)} \geq \frac{\sigma_{S^{d-1}}(S_h)^{1/p}}{100h^2}$ (*)

On the other hand, the maximal conjecture implies

$\|f_{2^{-h}}\|_p \leq c_\epsilon 2^{h\epsilon} \|f\|_p \leq c_\epsilon 2^{h\epsilon} (|J_h| 2^{-(h-1)d})^{1/p} \leq c_\epsilon 2^{h\epsilon} (|J_h| 2^{-hd})^{1/p}$

\Rightarrow Combining this with the upper bound (*) shows

$\frac{\sigma_{S^{d-1}}(S_h)}{100h^2} \leq c_\epsilon 2^{h\epsilon} 2^{h(\epsilon-d)} h^{2p} |J_h| \leq 2^{h(2\epsilon-d)} |J_h|$

$\Rightarrow \sum_{j \geq 0} r_j^{d-2\epsilon} = \sum_h \sum_{j \in J_h} r_j^{d-2\epsilon} \geq \sum_h 2^{-h(d-2\epsilon)} |J_h| \geq \sum_h \sigma_{S^{d-1}}(S_h) \geq \sigma(US_h) = 1$

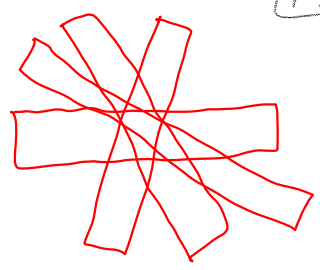
$\Rightarrow \sum r_j^{d-2\epsilon} \geq 1$ whenever $d-2\epsilon > 0$, i.e., $\sum r_j^\alpha \geq 1 \forall \alpha < d$

$\Rightarrow \dim_H(E) = d$ (since $\dim_H(E) \leq d$ trivially) $\Rightarrow \dim_H E \geq d$

(A slight generalization of this argument can be found in Sogge's book, Prop 9.1.5, see also Lemma 9.1.3)

We on

6.3 Restriction conjecture \Rightarrow Kahaya conjecture



To see this, we use the following duality argument

Lemma 6.11 Let $K \subset \mathbb{R}^d$ be such that it has the following property:

If $\{w_n\} \subset S^{d-1}$ is a maximal δ -separated set (in \mathbb{R}^d)
 and if $\sum_k y_k p^k \leq 1$, then for any choice of points $a_n \in \mathbb{R}^d$,
 we have $\| \sum_k y_k \mathbb{1}_{T_{w_n}^\delta(a_n)} \|_{p'} \leq A$.

$\| \sum \mathbb{1}_{T_w} \|_1$

$\ll \sum \| \mathbb{1}_{T_w} \|_p = N \frac{1}{\delta^{d-1}}$

Then, there is a bound $\| f_s^* \|_{p'} \leq A \| f \|_p$

For a metric set (X, d) , a subset $A \subset X$ is said to be maximal δ -separated if

- $\forall x \neq y$ in A , we have $d(x, y) \geq \delta$
- $\forall z \in X \setminus A \exists w \in A$ $d(z, w) < \delta$

$\| g \delta \|^1 \| g \|_q \leq \| g \|_q \dots$



\rightarrow here: δ -separated subset of S^{d-1} has cardinality $\leq \delta^{-(d-1)}$

$\| \sum \mathbb{1}_{T_w} y_n \|_{L^q}$

Pf Suppose $\{w_n\}_n$ is a maximal δ -separated subset of S^{d-1}
 Observe that if $|w - w'| < \delta$, then $f_s^*(w) = \sup_a \frac{1}{|T_w^\delta(a)|} \int_{T_w^\delta(a)} |f| \leq c f_s^*(w')$ since

to each $T_w^\delta(a)$ can be covered by a bounded number of tubes $T_{w'}^\delta(a')$.

\Rightarrow Using $\| f \|_{p'} = \sup_{\substack{g \in L^p \\ \|g\|_p=1}} | \langle g, f \rangle |$ with $\vec{f} = (f_n)_{n \in \mathbb{N}}$, $f_n = \delta^{(d-1)/p} |f_s^*(w_n)|$ shows
 $\vec{g} = (g_n)_{n \in \mathbb{N}}$, $g_n = y_n \delta^{(d-1)/p}$ satisfies above assumption

$\| f_s^* \|_p \leq \left(\sum_n \int_{B(w_n, \delta)} dw |f_s^*(w)|^p \right)^{1/p} \leq c \left(\sum_n |f_s^*(w_n)|^p \cdot |B_{w_n}(\delta)| \right)^{1/p}$
 $\leq c^p |f_s^*(w')|$ for $|w - w'| < \delta$; take $w' = w_n$

$\ll \sim c \| \vec{f} \|_{\ell^p} = c \sum_n \langle \vec{g}, \vec{f} \rangle = c \cdot \delta^{d-1} \sum_n y_n |f_s^*(w_n)|$

$= c \delta^{d-1} \sum_n y_n \cdot \frac{1}{|T_{w_n}^\delta(a_n)|} \int_{T_{w_n}^\delta(a_n)} |f|$ for some suitable $\{a_n\}$
 $\sim \delta^{1-d}$

$\sim c \int \left(\sum_n y_n \mathbb{1}_{T_{w_n}^\delta(a_n)} \right) |f| \stackrel{\text{Holder}}{\leq} c \| f \|_p \underbrace{\| \sum y_n \mathbb{1}_{T_{w_n}^\delta(a_n)} \|_{p'}}_{\leq A \text{ by assumption}} \leq c \cdot A \| f \|_p$



Thm 6.12 (Restriction \Rightarrow Kahya)

Suppose $\|f\|_{L^q(S^{d-1})} \leq \|f\|_{L^p}$ whenever $p < \frac{2d}{d+1}$ & $\frac{d+1}{p} \leq \frac{d-1}{q}$. Then $\|f\|_{L^q(S^{d-1})} \leq C \|f\|_{L^p}$

then for a maximal S -separated set $\omega \in S^{d-1}$ and a collection

~~$\{T_\omega^S(0)\}_{\omega \in \Omega}$, we have $\|\sum_{\omega \in \Omega} \mathbb{1}_{T_\omega^S(0)}\|_{L^{d/(d-1)}(\mathbb{R}^d)} \lesssim \delta^{-\epsilon} \left(\sum_{\omega \in \Omega} \mathbb{1}_{T_\omega^S(0)}\right)^{\frac{d-1}{d}}$~~

~~note $\left(\frac{d}{d-1}\right)' = d$~~

$p > \frac{2d}{d-1}$

By restriction, we know that $\|(g d\sigma)^\vee\|_{L^{2d/(d-1)}(\mathbb{R}^d)} \leq \|g\|_{L^{2d/(d-1)}(S)}$ barely fails,

all we need here \rightarrow

$\|(g d\sigma)^\vee\|_p \leq \|g\|_p, p > \frac{2d}{d-1}$

(recall exercise 4.4 on equivalent formulations of restriction)

but we do have the estimates $\|(g d\sigma)^\vee\|_{L^{2d/(d-1)}(B_{\delta^{-1}}(S^{-1}))} \lesssim \delta^{-\epsilon} \|g\|_{L^{2d/(d-1)}(S^{d-1})}$

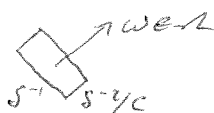
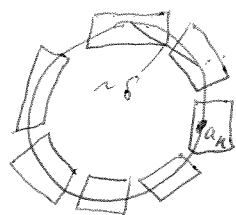
via restriction + the ϵ -removal lemma which states " $R_S(p \rightarrow p; \alpha) \Rightarrow R_S(q \rightarrow q)$ "

whenever $\frac{1}{q} > \frac{1}{p} + \frac{1}{\log 1/\alpha}$

$\|f\|_{L^q(S)} \lesssim \delta^{-\epsilon} \|f\|_{L^q(B_{\delta^{-1}}(S^{-1}))}$

the caps associated to the

Let $f = \sum_k \mathbb{1}_k \gamma_k$ where x are $S^{+1}x \dots x S^{+1}x \frac{1}{C} S^{+2}$ boxes with $C > 1$ so large that these boxes do not overlap if we arrange them along the S -separated subset of directions



$x_\omega = T_\omega^S(a_\omega) \cap S^{d-1}$
 T_ω^S dimensions $S^{+1}x \dots x S^{+1}x \frac{\delta^{+2}}{C}$



$\|(g d\sigma)^\vee\|_{L^p} \leq \|g\|_{L^p}$

\rightarrow all of the caps are Knapp examples

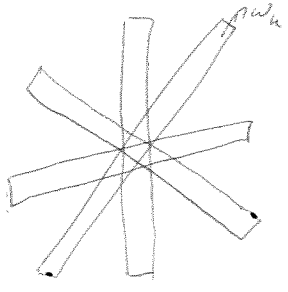
Now let's randomize these Knapp examples to exploit cancellation effects which reflect that the dual tubes are heavily compressed in position space.

$g = \sum_k \underset{\substack{\uparrow \\ \text{random signs}}}{\rho_k} \gamma_k \mathbb{1}_k$

since the caps are disjoint, we have

$\|g\|_{L^q(S^{d-1})}^q \sim \sum_k \int_k |\gamma_k|^q \sim \sum_k |\gamma_k|^q \cdot \delta^{-(d-1)}$

On the other hand $\widehat{f f d\sigma}(x) = \sum \rho_k \gamma_k \cdot \delta^{d-1} \chi_{\tau_k}$ where χ_{τ_k} are Schwartz fcts adapted to the dual tubes τ_k oriented along ω_k , of dimensions $S^{-1} \times (S^1)^{d-1}$ centered at origin, (at least if the f_k are real-valued.)



By the randomization, we can use Khintchine's inequality

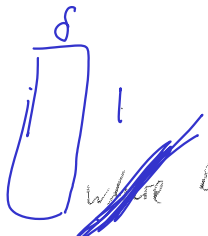
$$\left(\sum |a_n|^2\right)^{p/2} \sim \mathbb{E}_y \left| \sum y_n a_n \right|^p$$

$$\Rightarrow \mathbb{E}_y \left\| \left(\sum y_n \delta \right)^v \right\|_q^q \sim \mathbb{E} \int \left| \sum y_n y_n \delta^{d-1} \chi_{\tilde{z}_n} \right|^q$$

$\|g\|_q^q \rightarrow \delta$

$$\sim \int \left(\sum |y_n|^2 \delta^{2(d-1)} |\chi_{\tilde{z}_n}(x)|^2 \right)^{q/2}$$

$$\geq \delta^{q(d-1)} \int \left(\sum |y_n|^2 |\tilde{\chi}_{\tilde{z}_n}(x)|^2 \right)^{q/2}$$

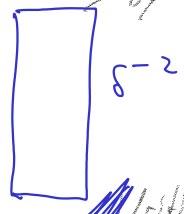


where $\tilde{\chi}_{\tilde{z}} = \chi_{\tilde{z}}|_{\tilde{z}}$, i.e., we remove all Schwartz tails

$$\rightarrow \left\| \sum_n y_n \chi_{T_{\omega_n}} \right\|_{p'} \leq \sqrt{d}$$

by restriction, we get for any $q > \frac{2d}{d-1}$ that

$$\Rightarrow \left\| \frac{1}{\delta} \right\|_q \leq A \left\| \frac{1}{\delta} \right\|_{p'}$$



$$\delta^{q(d-1)} \int \left(\sum_n |y_n|^2 |\chi_{\tilde{z}_n}(x)|^2 \right)^{q/2} \leq \sum |y_n|^q \delta^{-(d-1)}$$

Writing $z_n = y_n^2$ and $p' = q/2$, this is equivalent to the statement

"If $\delta^{d-1} \sum_n (z_n)^{p'} \leq 1$ then $\left\| \sum_n z_n \chi_{\tilde{z}_n} \right\|_{p'} \leq \delta^{-2(d-1)}$ for any $p' \geq \frac{d}{d-1}$ "

We now rescale the \tilde{z}_n ($\delta^{-1} \times \delta^{-1} \times \delta^{-2}$ -tubes) to T_n , i.e., $\delta \times \delta \times \delta$ -tubes, (\sim rescale $\times \delta^2$)

the ones, we dealt with earlier, in particular in Lemma 6.

\Rightarrow "If $\delta^{d-1} \sum_n (z_n)^{p'} \leq 1$, then $\left\| \sum_n z_n \chi_{T_n} \right\|_{p'} \leq \delta^{-2(d-1+d/p')}$ for $p' \geq \frac{d}{d-1}$ "

Observe that $d-1+\frac{d}{p'} > 0$ as $p' \geq \frac{d}{d-1}$, so for any $\epsilon > 0$ we showed

"If $\delta^{d-1} \sum_n (z_n)^{p'} \leq 1$, then $\left\| \sum_n z_n \chi_{T_n} \right\|_{p'} \leq \epsilon \delta^{-\epsilon}$ if p' is close to d "

\Rightarrow By Lemma 6.11 this implies that for any $\epsilon > 0$, we have $\left\| \frac{1}{\delta} \right\|_{p'} \leq \epsilon \delta^{-\epsilon} \left\| \frac{1}{\delta} \right\|_d$

provided $p < d$ is close enough to d . Interpolating this with the $\left\| \frac{1}{\delta} \right\|_d \leq \left\| \frac{1}{\delta} \right\|_d$ bound we obtain $\left\| \frac{1}{\delta} \right\|_d \leq \epsilon \delta^{-\epsilon} \left\| \frac{1}{\delta} \right\|_d$, as desired

reflects extreme compression of z_n . If $z_n = 1$ then LHS satisfied and RHS reads $\left\| \sum_n \chi_{z_n} \right\|_{p'} \leq \delta^{-2(d-1)}$

would be justified for disjoint tubes

worse by a factor $\delta^{2-1/d}$

each of z_n has $L^{p'}$ -mass $\sim \delta^{-(d-1)/p'}$

$$= 2d - \frac{1}{d}$$

triangle inequality would give $\delta^{-\left(\frac{d+1}{p'} + d-1\right)} = \delta^{-(2d-\frac{1}{d})}$

of tubes $\sim \delta^{-(d-1)}$

6.4 Proof of Kahya maximal conjecture in $d=2$ (Cordoba, Bourgain) 1977 1981

(15)
 $\delta^{-\epsilon} \gg \sqrt{\log \frac{1}{\delta}}$

Thm 6.13 If $d=2$, then $\|f\delta^*\|_{L^2(\mathbb{S}^1)} \lesssim \sqrt{\log \frac{1}{\delta}} \|f\|_{L^2(\mathbb{R}^2)}$

$f\delta^*(\omega) = \frac{1}{|\mathbb{T}_\omega^\delta(\omega)|} \int_{\mathbb{T}_\omega^\delta(\omega)} \frac{f}{\delta^{\frac{d-1}{2}}}$



Proof of Thm 6.13 due to Bourgain (using Fourier analysis)

Wlog suppose $f \geq 0$ and let $p_\delta^\omega(x) := \frac{1}{2\delta} \mathbb{1}_{\mathbb{T}_\omega^\delta(x)}$, then $f\delta^*(\omega) = \sup_{a \in \mathbb{R}^2} (p_\delta^\omega * f)(a)$.

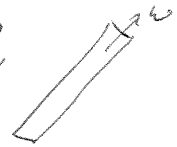
Let $0 \leq \phi \in \mathcal{S}(\mathbb{R})$ with $\phi(x) \geq 1$ for $|x| \leq 1$ and $\hat{\phi}$ compactly supported

Let $\psi(x) = \phi(x_1) \cdot \frac{1}{\delta} \phi(\frac{x_2}{\delta})$ and note $\psi \geq p_\delta^\omega$ if $\omega = \hat{e}_1$, i.e., $f\delta^*(\hat{e}_1) \leq \sup_a (\psi * f)(a)$

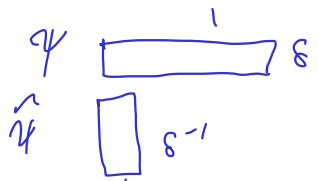
Analogously, $f\delta^*(\omega) \leq \sup (\psi_\omega * f)(a)$ where $\psi_\omega = \psi \circ R_\omega$ for an appropriate $R_\omega \in SO(2)$

$\Rightarrow f\delta^*(\omega) \leq \|\psi_\omega * f\|_\infty \leq \|\hat{\psi}_\omega \hat{f}\|_1 = \int |\hat{\psi}_\omega(\xi)| |\hat{f}| \leq \left(\int |\hat{\psi}_\omega(\xi)|^2 \langle \xi \rangle^2 d\xi \right)^{1/2} \left(\int d\xi \frac{|\hat{f}|^2}{\langle \xi \rangle^2} \right)^{1/2}$

To compute the second factor we use $\hat{\psi}_\omega = \hat{\psi} \circ R_\omega$ and $\hat{\psi} = \hat{\phi}(k_1) \hat{\phi}(\delta k_2)$ s.t. $|\hat{\psi}_\omega| \leq 1$ and $\hat{\psi}$ is supported on a rotated $1 \times \delta^{-1}$ rectangle box P_ω



$\Rightarrow \int \frac{|\hat{\psi}_\omega(\xi)|}{\langle \xi \rangle^2} \leq \int_{P_\omega} \frac{d\xi}{\langle \xi \rangle^2} \sim \int_{\xi_2}^{\delta^{-1}} \frac{d\xi_2}{\xi_2} \sim \log \frac{1}{\delta}$



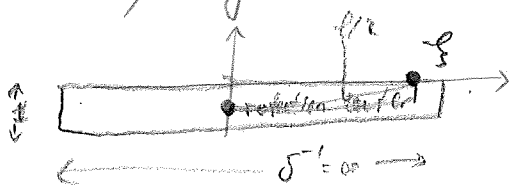
$\Rightarrow \|f\delta^*\|_2^2 \lesssim \log \frac{1}{\delta} \int \int d\xi \langle \xi \rangle^2 |\hat{f}|^2 |\hat{\psi}_\omega(\xi)| d\omega$ (*)

Claim: For fixed $\xi \in \mathbb{R}^d$, ~~the set~~ $\{ \omega \in \mathbb{S}^{d-1} : |\hat{\psi}_\omega(\xi)| > 0 \} \leq \langle \xi \rangle^{-1} \forall \delta > 0$

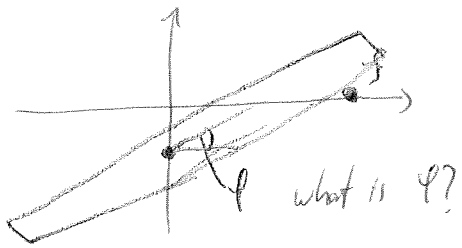
Believing the claim shortly shows (*) $\leq \log \frac{1}{\delta} \int d\xi |\hat{f}|^2 \square$

Proof of claim By symmetry it suffices to consider $d=2$, wlog. $\xi = (\xi_1, 0)$

with $|\xi_1| \gg 1000$, say. (Independently of δ , although we still keep in mind that ψ_ω is \sim constant on the $1 \times \delta^{-1}$ tube. In fact we may even assume that it's infinitely elongated



now let's rotate and see how $\{ \omega \in \mathbb{S}^1 : |\hat{\psi}_\omega(\xi_1, 0)| > 0 \} \leq \int_{\mathbb{S}^1} \mathbb{1}_{\mathbb{T}_\omega^\delta(\xi)} \sim \psi \sim \frac{1}{|\xi|}$ gets smaller



$\sin \frac{\theta}{2} = \frac{1/2}{|\xi|} \sim \sin \theta \sim \delta/R \Rightarrow \theta \sim |\xi|^{-1}$

Remark For $d \geq 3$ the $\log V_S$ in $\int \frac{|\hat{w}(s)|}{(s^d)} \leq \log V_S$ becomes $s^{-(d-2)}$ and hence

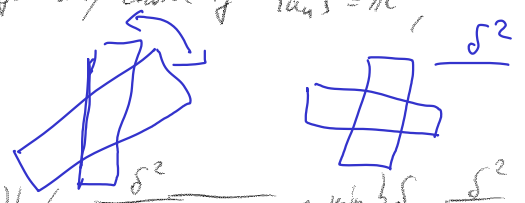
$$\|f_S\|_{L^2(s^{d-1})} \leq s^{-(d-2)/2} \|f\|_{L^2(\mathbb{R}^d)}$$

which is the best possible L^2 bound

Proof of Thm 6.13 due to Cordoba (geometric)

By Lemma 6.11 it suffices to prove that for any sequence $\{y_n\}$ with $\sum y_n^2 = 1$ and any maximal δ -separated subset $\{w_n\} \subseteq S'$, we have for any choice of $\{a_n\} \subseteq \mathbb{R}^d$,

$$\| \sum_n y_n \mathbb{1}_{T_{w_n}^\delta(a_n)} \|_2 \leq \sqrt{\log V_S}$$



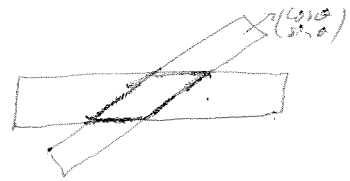
HW The relevant geometric fact is $|T_{w_n}^\delta(a) \cap T_{w_e}^\delta(b)| \leq \frac{\delta^2}{\delta + |w_n - w_e|}$ $\{ \min \{ \delta, \frac{\delta^2}{|w_n - w_e|} \}$

Clearly it suffices to consider $a=b$, $e_2=e_1=(1,0)$, and that the angle θ between $w_n = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ and e_1 is $\leq \pi/2$. Since $|T_{w_n}^\delta(a)| = \delta$, the first bound is clear.

Now suppose $|w_n - e_1| = \sqrt{2} \sqrt{1 - \cos \theta} \geq \delta$ (otherwise min will be attained by δ)
 $\Rightarrow \theta > \sqrt{2} \delta$ by using $\cos \theta < 1 - \frac{\theta^2}{4}$

In this case $\sin \theta > \frac{\theta}{2} > \delta/2$ and we have $\frac{\delta}{2 \sin \theta} < 1$. Using $\cos \theta \geq 1 - \theta^2$ and the formula for the surface area of a parallelogram, we get

$$|T_{w_n}^\delta \cap T_{w_e}^\delta| = 4 \frac{\delta}{2} \cdot \frac{\delta}{2 \sin \theta} \leq \frac{2\delta^2}{\theta} \leq \frac{2\sqrt{2}\delta^2}{\sqrt{2}\sqrt{1-\cos \theta}} = \frac{2\sqrt{2}\delta^2}{|w_n - w_e|} \quad \square$$



$$\delta \wedge \frac{\delta^2}{|w_n - w_e|}$$

Using this geometric observation allows us to estimate

$$\begin{aligned} \| \sum_n y_n \mathbb{1}_{T_{w_n}^\delta} \|_2^2 &= \sum_{k,l} y_k y_l |T_{w_k}^\delta(a_k) \cap T_{w_l}^\delta(a_l)| \leq \sum_{k,l} y_k y_l \frac{\delta^2}{\delta + |w_k - w_l|} \\ &= \sum_k \sqrt{\delta} y_k \sqrt{\delta} y_l \underbrace{\frac{\delta}{\delta + |w_k - w_l|}}_{=K_{kl}} \leq \| \sqrt{\delta} y_k \|_{l^2} \| \sqrt{\delta} y_l \|_{l^2} \leq \ln \frac{1}{\delta} \| \sqrt{\delta} y_k \|_{l^2}^2 \\ &= K_{kl} \quad \quad \quad l^2 \text{ means summation over } l \end{aligned}$$

Now recall that the set $\{w_n\}$ is δ -separated, i.e., for fixed k there are at most δ^{-1} many summands in the l -summation. Moreover, since the angle between w_k and w_l is given by $\sim \delta |k-l|$, ($w_k \cdot w_l = \cos \theta$)

we have $|w_k - w_l| = \sqrt{2} \sqrt{1 - \cos(\delta |k-l|)} \geq \sqrt{\frac{2}{100}} \delta |k-l|$ for $\delta |k-l| < 1$



$$\Rightarrow \sup_k \sum_l \frac{\delta}{\delta + |w_k - w_l|} \leq \sum_{l < k/\delta} \frac{\delta}{\delta + \delta l} \sim \log V_S \rightarrow \text{by a symmetric Schur test } (K_{kl} = K_{lk}),$$

we are done

6.5 Can Kahya help to prove restriction? (Bourgain 1991) (17)

Thm 6.14 Suppose we have an estimate $(*) \|\sum_j \frac{1}{\omega_j} f_j\|_{q'} \lesssim \delta^{-(\frac{d}{q}-1+\epsilon)}$ for any

given $\epsilon > 0$ and some fixed $q > 2$. Then $\|(f d\sigma)^\vee\|_p \lesssim_p \|f\|_{L^\infty(S^{d-1})}$ for some $p < \frac{2(d+1)}{d-1}$

recall equivalent formulations of restriction (ex 6.4)

Remark (1) The geometrical statement corresponding to (*) is that Besicovitch sets in \mathbb{R}^d have Hausdorff dimension at least q (see, e.g. Sogge 9.1.5)

We will only sketch the proof below for $d=3$ (see Wolff Thm 10.6). Recall that in this case we already know the bounds $\|(f d\sigma)^\vee\|_{L^p(\mathbb{R}^3)} \lesssim \|f\|_{L^2}$ (Tomas-Stein) and

$$\|(f d\sigma)^\vee\|_{L^2(B_0(R))} \lesssim R^{1/2} \|f\|_{L^2} \quad (\text{exercise!})$$

$\alpha = d-1=2$

$$\Rightarrow \text{interpolation gives } \|(f d\sigma)^\vee\|_{L^p(B_0(R))} \lesssim R^{2/p-1/2} \|f\|_{L^2(S^2)}, \quad 2 < p < 4$$

We now show that the power $\frac{2}{p} - \frac{1}{2}$ of R can be lowered by ϵ , if $L^2(S^2)$ norm on RHS is replaced by L^∞ norm

Prop. 6.15 Let $d=3$, $2 < p < 4$ and assume $\|\sum_j \frac{1}{\omega_j} f_j\|_{q'} \lesssim \delta^{-(\frac{d}{q}-1+\epsilon)}$ holds for some $q > 2$

$$\text{Then } \|(f d\sigma)^\vee\|_{L^p(B_0(R))} \lesssim R^{\alpha(p)} \|f\|_{L^\infty(S^{d-1})} \text{ where } \alpha(p) < \frac{2}{p} - \frac{1}{2}$$

\Rightarrow This implies Thm 6.14 for all p st. $\frac{2}{p} - \frac{1}{2} \leq \alpha$, i.e., $p < 4$

Remark Similar type of lemma due to Tao "epsilon-removal lemma" (1999) (see Appendix A in notes)

Heuristic proof of Prop 6.15

By homogeneity we may assume $\|f\|_{L^\infty(S^{d-1})} = 1$ and let $S = \mathbb{R}^{-1}$ in above mention

Now cover S^2 by spherical caps $S_j = \{\omega \in S^2 : |1 - \omega \cdot e_j| \leq \delta\}$ where the $\{e_j\}$ shall form a maximal 2δ separated sub-set of S^2 . Then decompose

$$f = \sum f_j, \quad \text{supp } f_j \subseteq S_j \quad \text{and let } G_j = (f_j d\sigma)^\vee, \quad G = (f d\sigma)^\vee = \sum G_j.$$

By the uncertainty principle, G_j roughly constant on dual tubes of dim $\mathbb{R}^{\frac{1}{2}} \times \mathbb{R}^{\frac{1}{2}} \times \mathbb{R}$ pointing in direction e_j . ~~tho~~ We will ~~not~~ now cheat and neglect their Schwartz tails

Now cover $B_0(R)$ with disjoint cubes Q of side length \sqrt{R} . For each Q we denote by $N(Q)$ the number of tubes τ_j which intersect it.

Note that $G|_Q = \sum_j G_j|_Q$ where the sum is only over those j for which $\tau_j \cap Q \neq \emptyset$. Using this and the family of estimates $\| (f|_Q)^\vee \|_{L^p(B_0(R))} \lesssim R^{\frac{2}{p}-\frac{1}{2}} \|f\|_{L^2}$ obtained from interpolation, we can estimate for $2 \leq p \leq 4$,

$$\|G\|_{L^p(Q)} \lesssim R^{\frac{1}{2}(\frac{2}{p}-\frac{1}{2})} \left\| \sum_{j: \tau_j \cap Q \neq \emptyset} f_j \right\|_{L^2(S^2)} \lesssim R^{\frac{1}{2}(\frac{2}{p}-\frac{1}{2})} \sqrt{N(Q) \cdot |S^2|} \quad (\text{any } \ell)$$

$$\stackrel{R=\delta}{\sim} \delta^{\frac{3}{4}-\frac{1}{p}} \sqrt{N(Q)}$$

\Rightarrow Summing over Q yields

$$\|G\|_{L^p(B_0(R))}^p \lesssim \delta^{\frac{3p}{4}-1} \sum_Q N(Q)^{p/2} \sim \delta^{\frac{3p}{4}+\frac{1}{2}} \left\| \sum_j \mathbb{1}_{\tau_j} \right\|_{p/2}^{p/2}$$

where we used $\| \sum \mathbb{1}_{\tau_j} \|_{p/2}^{p/2} = \sum N(Q)^{p/2} \cdot |Q| = \delta^{-3/2} \sum N(Q)^{p/2}$

Now let $p=2q'$ where q' is as in the argument in unimed bound in Thm 6.14 and assume that p is sufficiently close to 4 (otherwise interpolate between unimed. of Thm 6.14 and $\|f_S^*\|_2 \lesssim \delta^{-(d-2)/2} \|f\|_2$ from proof of Thm 6.13 due to Bourgain for $d=3$) to get p close to 4.

By $\| \sum \mathbb{1}_{T_{w_j}^{\sqrt{R}}} \|_{q'} \lesssim \delta^{-\frac{1}{2}(\frac{d}{q'}-1+\epsilon)} \| \sum \mathbb{1}_{T_{w_j}^{\sqrt{R}}} \|_{q'}$ observe the \sqrt{R} !

Rescaling by δ^{-1} (i.e., replacing $T_{w_j}^{\sqrt{R}}$ by τ_j with dimensions $R^{\frac{1}{2}} \times R^{\frac{1}{2}} \times R^1 = \delta^{-1/2} \times \delta^{-1/2} \times \delta^{-1}$) gives $\| \sum \mathbb{1}_{\tau_j} \|_{q'} \lesssim \delta^{-\frac{1}{2}(\frac{d}{q'}-1+\epsilon)} \cdot \delta^{-3/4'} = \delta^{-1-3/p-\epsilon}$

Combining this with $\|G\|_{L^p(B_0(R))} \lesssim \delta^{3/4+1/2p} \| \sum \mathbb{1}_{\tau_j} \|_{p/2}^{1/2}$ shows

$$\|G\|_{L^p(B_0(R))}^p \lesssim \delta^{\frac{3p}{4}+\frac{1}{2}-p-\frac{3}{2}-\epsilon p/2} = \delta^{p/4-1-\epsilon}$$

$$\Rightarrow \| (f|_Q)^\vee \|_{L^p(B_0(R))} \lesssim \delta^{1/4-1/p+\epsilon} = R^{-\frac{1}{4}+\frac{1}{p}+\epsilon}$$

ob if $p < 4$

$R^{-\frac{1}{4}+\frac{1}{p}}$ beats the $R^{\frac{2}{p}-\frac{1}{2}}$ from interpol.

if $p < 4$ □