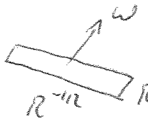
Remarks on locally constant lemma

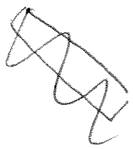
$|f(z)| \lesssim \|f\|_{L^1(\phi_{E^*}\,dx)}$ whenever $\operatorname{supp}\hat{f} \subseteq E$, $z \in E^*$ or any translate thereof.

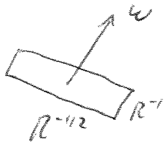
Suppose for simplicity that $E \overset{\theta_\omega}{|||}$ is a rotated $R^{-\frac12}\times\dots\times R^{-\frac12}\times R^{-1}$ rectangle, oriented along $\omega\in S^{d-1}$.



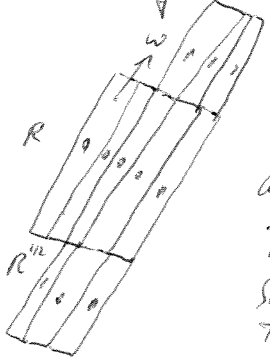
Then, if f is real-valued, we may use stationary phase to infer that f is a Schwartz fct focussing on E^*, ~~focused~~ centered at the origin.

If $\hat{f}$ is not real-valued, we expand $\hat{f}$ in a Fourier series with "frequencies" running through the centers of all translates of E^* that tile $\mathbb{R}^d$.





frequency space



Let T_ω^a denote that dual rectangle centered at $a\in\mathbb{R}^d$ with direction ω and collect all of them in the family $\mathbb{T}_\omega \equiv \{T_\omega^a\}_a$ tiling $\mathbb{R}^d$. Sometimes we write "$a\in\mathbb{T}_\omega$" to indicate $T_\omega^a\in\mathbb{T}_\omega$.

To make this more precise, let $\varphi\in C_c^\infty([-\frac12,\frac12]^d)$ and write $\hat{f} = \varphi\circ T$ where $T = DR$ where $R\in SO(d)$ which rotates [rectangle] $\mapsto$ [rectangle], i.e., $R\omega = e_d$, and $D = \operatorname{diag}(R^{1/2},\dots,R^{1/2},R)$ scales the rectangle to the unit box $\underset{R^{-1/2}}{\square}{}_{R^{-1}} \mapsto \square_1^1$

~~Let $T_{\omega,a}$~~

<u>Goal</u> Expand $\hat{f}(\xi) = \overset{\text{uniform convergence whenever } \hat f\in L^1(\mathbb{T}^d) \text{ and continuously differentiable}}{\sum_{a\in\mathbb{T}_\omega}}{}' f_\omega(a)\, e^{2\pi i a\cdot\xi}\,\mathbb{1}_{\theta_\omega}(\xi)$. What are the $f_\omega(a)$?

We prepare for $a\in\mathbb{T}_\omega$.

$$\langle e^{2\pi i\langle a,\cdot\rangle}, f\rangle = \int d\xi\, (\varphi\circ T)(\xi)\, e^{2\pi i a\cdot\xi} \cancel{= \frac{1}{|\det T|}\int d\xi\, \varphi(\xi)}$$

$$= \frac{1}{|\det T|}\int d\xi\, \varphi(\xi)\, e^{2\pi i\langle T^{-t}a,\,\xi\rangle}$$

$$= \frac{1}{|\det T|}\check{\varphi}(T^{-t}a) \quad \text{where } T^{-t}a\in\mathbb{Z}^d \text{ whenever } a\in\mathbb{T}_\omega$$

~~$\Rightarrow \sum_{w\in\mathbb{T}_\omega} |\det T|\langle e^{2\pi i\langle a,\cdot\rangle}, f\rangle = \varphi($~~

$$\Rightarrow \sum_{w\in\mathbb{T}_\omega} |\det T|\langle e^{2\pi i\langle a,\cdot\rangle}, f\rangle\, e^{2\pi i\langle T^{-t}a,\,\xi\rangle} = \varphi(\xi)$$

$\Rightarrow$ Replacing $\xi\mapsto T\xi$, we obtain $\hat{f}(\xi) = (\varphi\circ T)(\xi) = \sum_{a\in\mathbb{T}_\omega} |\det T|\langle e^{2\pi i\langle a,\cdot\rangle}, f\rangle\, e^{2\pi i a\cdot\xi}\,\mathbb{1}_{\theta_\omega}(\xi)$

focussed at T_ω^a

$$\rightarrow f(x) = \sum_{a\in\mathbb{T}_\omega}\langle e^{2\pi i\langle a,\cdot\rangle}, f\rangle\, \chi_{T_\omega}(x-a)$$

$R^{(d+1)/2}\mathcal{F}(\mathbb{1}_{\theta_\omega})(x)$ with $\|\cdot\|_\infty \sim 1$.

Moreover, we have a Plancherel similarity

$$\|\hat{f}\|_2^2 = |\det T|^2 \sum_{a,b\in\pi_\omega} \langle e^{2\pi i\langle a,\cdot\rangle}, f\rangle\langle f, e^{2\pi i\langle b,\cdot\rangle}\rangle \int d\xi \; e^{2\pi i \xi(a-b)} \mathbb{1}_{\Theta_\omega}(\xi)^2 \left(\delta_{a,b} + 1 - \delta_{a,b}\right)$$

$$= |\det T| \sum_{a\in\pi_\omega} |\langle e^{2\pi i\langle a,\cdot\rangle}, f\rangle|^2 + \sum_{a\neq b\in\pi_\omega} \langle e^{2\pi i\langle a,\cdot\rangle}, f\rangle\langle f, e^{2\pi i\langle b,\cdot\rangle}\rangle \int_{\mathbb{R}^d} dy\; \mathcal{X}_{T_\omega}(a-b-y)\mathcal{X}_{T_\omega}(y)$$

$$\sim R^{\frac{d+1}{2}} \sum_{a\in\pi_\omega} |\langle e^{2\pi i\langle a,\cdot\rangle}, \hat{f}\rangle|^2$$

by Parseval, or almost orthogonality of the $\mathcal{X}_{T_\omega}$ (e.g. by Cauchy-Schwarz for $\sum_{a,b}$, i.e.

~~and~~ $\cancel{\|\mathcal{X}_{T_\omega}\|_\infty \lesssim 1}$, ~~i.e.,~~ $\cancel{\sum_{a,b} f_\omega(a) f_\omega(b) \underbrace{\int \mathcal{X}_{T_\omega}(y)\,dy}_{\sim R^{(d+1)/2}} \mathcal{X}_{T_\omega}(a-b-y)}$

$$\cancel{\underbrace{\int dy \sum_{a,b} f_\omega(a)\overline{f_\omega(b)}\, \mathcal{X}_{T_\omega}(a-y)\mathcal{X}_{T_\omega}(b-y)}_{\int dy \sum_a f_\omega(a)\mathcal{X}_{T_\omega}(a-y) \sum_b \overline{f_\omega(b)\mathcal{X}_{T_\omega}(b-y)}}} < \int dy \left|\sum_{a\in\pi} f_\omega(a)\, \mathcal{X}_{T_\omega}(a-y)\right|^2$$

$$\cancel{\overset{\text{Plancherel}}{=} \int d\xi \left|\sum_a f_\omega(a)\, \mathbb{1}_{\Theta_\omega}(\xi)\right|^2 \cdot R^{d+1}}$$

$$\cancel{\sim \sum_a |f_\omega(a)|^2 R^{(d+1)/2}} \quad \cancel{\int \mathbb{1}_{\Theta_\omega} = R^{-(d+1)/2}} \;)$$

$\mathcal{X}_{T_\omega} * \mathcal{X}_{T_\omega}$ ... Schwartz fct adapted to, say doubly dilated T_ω with $\|\mathcal{X}_{T_\omega} * \mathcal{X}_{T_\omega}\|_\infty \sim R^{\frac{d+1}{2}}$

$$\rightarrow \sum_{a\neq b} |\underbrace{f_\omega(a)}_{\equiv \langle e^{2\pi i\langle a,\cdot\rangle}, \hat{f}\rangle}|\,|f_\omega(b)|\, (\mathcal{X}_{T_\omega} * \mathcal{X}_{T_\omega})(a-b)$$

$$\lesssim_N \sum_{a\neq b} |f_\omega(a)|\,|f_\omega(b)|\, (1+|a-b|)^{-N} \lesssim \sum_{a\in\pi_\omega} |f_\omega(a)|^2 = \sum_{a\in\pi_\omega} |\langle e^{2\pi i\langle a,\cdot\rangle}, \hat{f}\rangle|^2$$

Small note on wave packet decomposition of solution of free Schrödinger with initial data having compact frequency support. 1

Goal: given $f \in \mathcal{S}(\mathbb{R}^{d-1})$ with $\operatorname{supp} \hat{f} \subseteq P^{d-1} = \{(\xi, \xi^2),\ \xi \in [-\frac{1}{2}, \frac{1}{2}]^{d-1}\}$, how does $\mathcal{F}_s^* f$ look like?

We begin with a frequency decomposition
Let $\Upsilon \in C_c^\infty([-2,2]^{d-1})$ s.t. $\sum_{j \in \mathbb{Z}^{d-1}} \Upsilon(\xi - j) = 1$. Now let's refine the mesh, i.e., let $R \gg 1$ and consider the lattice $R^{-1/2}\mathbb{Z}^{d-1}$. Then $\sum_{|j| \leq R^{1/2}} \Upsilon(R^{1/2}\xi - j) \sim \mathbb{1}_{|\xi| \leq 1}$ where $\Upsilon(R^{1/2}\xi - j)$ roughly equals an indicator function on the cube $\omega := c_\omega + [\frac{1}{\sqrt{R}}, \frac{1}{\sqrt{R}}]^{d-1}$ with $c_\omega = R^{-1/2} j \in R^{-1/2}\mathbb{Z}^{d-1}$, i.e., the c_ω denote the centers of the cubes with side lengths $\cancel{[\frac{1}{\sqrt{R}}, \frac{1}{\sqrt{R}}]}$ $\frac{2}{\sqrt{R}}$ and are roughly $R^{-1/2}$-separated from each other. We collect the centers c_ω in the $R^{-1/2}$-net $\Omega_R \subseteq R^{-1/2}\mathbb{Z}^{d-1}$.

Then, as in the "refined analysis" of the locally constant lemma, it is reasonable to decompose each $f\mathbb{1}_\omega$ in a Fourier series

$$\rightarrow f(\xi) = \sum_{c_\omega \in \Omega_R} \sum_{c_q \in Q_R} R^{\frac{d-1}{2}} \langle e^{2\pi i \langle c_q, \cdot \rangle}, f\mathbb{1}_\omega \rangle e^{2\pi i c_q \cdot \xi} \Upsilon(R^{\frac{1}{2}}(\xi - c_\omega))$$

$$\equiv \sum_{c_\omega \in \Omega_R} \sum_{c_q \in Q_R} \langle e^{2\pi i \langle c_q, \cdot \rangle}, f\mathbb{1}_\omega \rangle e^{2\pi i c_q \cdot c_\omega} \Upsilon_{q,\omega}(\xi)$$

where $c_q \in R^{1/2}\mathbb{Z}^{d-1}$ denote the center of the dual cubes $c_q + [-\sqrt{R}, \sqrt{R}]^{d-1}$ which are $\sim R^{1/2}$-separated from each other, and collected in the $R^{1/2}$-net Q_R.
Moreover, we defined $\Upsilon_{q,\omega}(\xi) = R^{\frac{d-1}{2}} e^{2\pi i c_q \cdot (\xi - c_\omega)} \Upsilon(R^{1/2}(\xi - c_\omega))$ which obeys $\|\Upsilon_{q,\omega}\|_2^2 \sim R^{\frac{d-1}{2}}$

Note that, since the frequency cubes ω overlap at most $O(1)$ many times, and by similar arguments as in the locally constant lemma in position space, we have the almost orthogonality

$$\|\sum_{q,\omega} w_{q,\omega} \Upsilon_{q,\omega}\|_2 \sim R^{\frac{d-1}{4}} \Big(\sum_{q,\omega} |w_{q,\omega}|^2\Big)^{1/2}, \quad (w_{q,\omega})_{\substack{q \in Q_R \\ \omega \in \Omega_R}} \subseteq \mathbb{C},$$

such as $w_{q,\omega} = w_q \delta_{\omega,\omega_0}$.

Now to understand $\mathcal{F}_s^* f$, we need to analyze

$$\phi_T(x) := \mathcal{F}_s^* \Upsilon_{q,\omega}(x) = e^{-2\pi i x \cdot (c_\omega, c_\omega^2)} \int d\eta\, \Upsilon_{q,\omega}(\eta) e^{2\pi i \varphi_{x,q,\omega}(\eta)}$$

with $\varphi_{x,q,\omega}(\eta) = \eta \cdot \frac{x' - c_q + 2c_\omega \cdot x_d}{\sqrt{R}} - \eta^2 \cdot \frac{x_d}{R}$

$\nabla_x \varphi: \begin{pmatrix} \eta/\sqrt{R} \\ 2c_\omega \frac{\eta}{\sqrt{R}} - \frac{\eta^2}{R} \end{pmatrix} \sim \begin{pmatrix} 1 \\ 2c_\omega \end{pmatrix} \frac{\eta}{\sqrt{R}}$

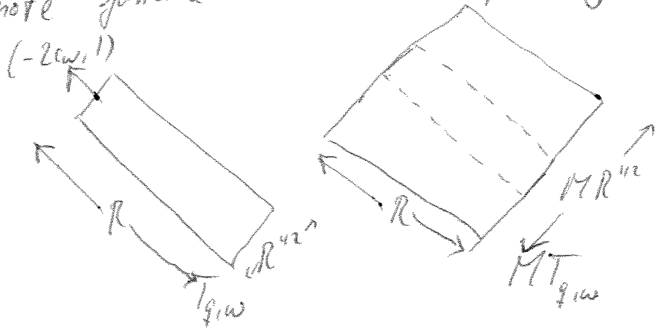
$\nabla_\eta \varphi \overset{!}{=} 0 \leftrightarrow x' - c_q + 2c_\omega x_d = \cancel{\eta}\, x_d / \sqrt{R}$

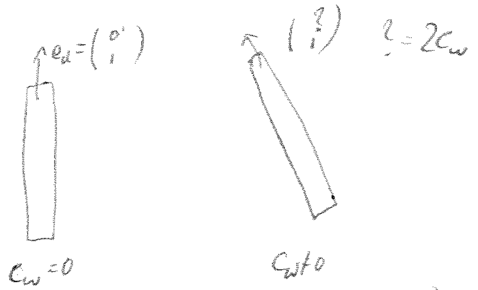
By stationary phase, we already anticipate that $\mathcal{F}_S^* \Psi_{q,\omega}$ will decay rapidly away from points where $\varphi \neq 0$, $\{\nabla \varphi = 0\}$, i.e., $\sim |x' - c_q + 2c_\omega x_d| < \sqrt{R}$, $|x_d| < R$
where potentially also

To make this more precise, define

$T_{q,\omega} = \{ x = (x', x_d) \in \mathbb{R}^d : |x' - c_q + 2c_\omega x_d| < \sqrt{R}, |x_d| < R \}$ which is a $R^{1/2} \times .. R^{\frac{1}{2}} \times R$-tube with center at $(c_q, 0)$, pointing in direction $\begin{pmatrix} -2c_\omega \\ 1 \end{pmatrix}$ (so that numerator vanishes for given x_d, here $x_d = 1$). The collection of all $T_{q,\omega}$ tiling $\mathbb{R}^d$ will be bundled in the family $\mathbb{T}_\omega$ for fixed frequency cube ω. The collection of all $\mathbb{T}_\omega$ for all ω is denoted by $\mathbb{T}$.

Moreover, for $M > 1$, let $MT_{q,\omega} = \{ x = (x', x_d) \in \mathbb{R}^d : |x' - c_\omega + 2c_\omega x_d| < MR^{1/2}, |x_d| < R \}$ denote fattened tube, where fattening only occurs around the central axis





($d = 2$, easy: if $x_d > 0$, $c_\omega = \begin{pmatrix} c_{\omega,1} \\ 0 \end{pmatrix}$, then x' must get smaller $\to$ tilt to left

<u>Thm (wave packet decomposition)</u>

Let $f \in C^\infty([-1,1]^{d-1})$, then there is a decomposition

$$f = \sum_{T \in \mathbb{T}} f_T \quad \text{with } \operatorname{supp} f_T \subseteq \omega_T \text{ for some } \omega_T = c_{\omega,T} + \left[-\frac{1}{\sqrt{R}}, \frac{1}{\sqrt{R}}\right]^{d-1}$$

with $c_{\omega,T} \in R^{-1/2} \Lambda_R \subseteq R^{-1/2} \mathbb{Z}^d$. Let $\mathcal{F}_S^* f_T = \underset{\in \mathbb{C}}{a_T} \varphi_T$ so that $\mathcal{F}_S^* f = \sum_{T \in \mathbb{T}} a_T \varphi_T$.

~~Let $\mathcal{F}_S^* f_T = a_T \varphi_T$ with $a_T \in \mathbb{C}$~~

Then 1) $\|\varphi_T\|_\infty \lesssim 1$, $\|\varphi_T\|_2 \lesssim R^{(d+1)/4}$, $\|\varphi_T\|_{L^\infty(\mathbb{R}^{d-1} \times [-R,R] \setminus MT)} \lesssim_k M^{-k}$, $M > 1$, $k \geq 0$

$\operatorname{supp} \widehat{\varphi_T} \subseteq \{(\xi, \xi^2) : \xi \in \omega_T\}$,

2) $\|f 1_\omega\|_2^2 \sim R^{(d-1)/2} \sum_{T \in \mathbb{T}_\omega} |a_T|^2$, $\|f\|_2^2 \sim R^{(d-1)/2} \sum_{T \in \mathbb{T}} |a_T|^2$

3) $\|f\|_2^2 \sim \sum_{T \in \mathbb{T}} \|f_T\|^2$

In particular, the choice $a_T = e^{2\pi i c_q \cdot c_\omega} \langle e^{2\pi i \langle c_q, \cdot \rangle}, f 1_\omega \rangle$ and $f_T = a_T \Psi_{q,\omega}$ is admissible.

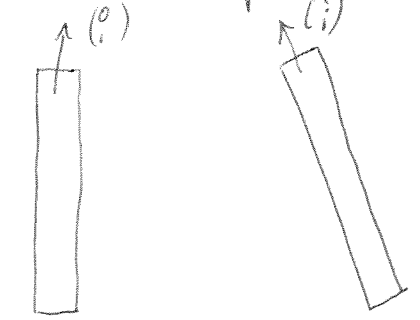
Pf Taking $a_T = e^{2\pi i c_q \cdot c_\omega} \langle e^{2\pi i \langle c_q, \cdot \rangle}, f\Delta_\omega \rangle$ and $f_T = a_T \Upsilon_{q,\omega}$ as indicated,

- then $\|\phi_T\|_{L^\infty(\mathbb{R}^{d-1}\times[-R,R]\setminus MT)} \lesssim_N M^{-N}$ $M \geq 1$, $N \geq 1$ follows from stationary phase arguments since $\cancel{\int_{y \in supp\,\Upsilon,\ x \in \mathbb{R}^{d-1}\times[-R,R]\setminus MT} |\nabla_y \psi_{x,q,\omega}(y)|}$ $\inf_{y \in supp\,\Upsilon,\ x \in (\mathbb{R}^{d-1}\times[-R,R])\setminus MT} |\nabla_y \psi_{x,q,\omega}(y)| \gtrsim M$

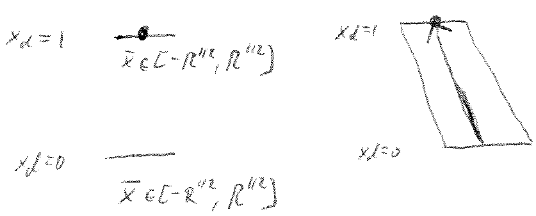
- $\|\phi_T\|_{L^\infty} \leq 1$ is obvious
- $\|f\Delta_\omega\|_2^2 \sim R^{(d-1)/2} \sum_{T\in\mathbb{T}_\omega} |a_T|^2$ follows from almost orthogonality of the $\Upsilon_{q,\omega}$ (or Parseval's identity) and since the cubes ω only finitely overlap

  and $\|f\|_2^2 \sim R^{(d-1)/2} \sum_{T\in\mathbb{T}} |a_T|^2$
- $\|f\|_2^2 \sim \sum_{T\in\mathbb{T}_\omega} |f_T|^2$ follows from $\|\Upsilon_{q,\omega}\|_2^2 \sim R^{(d-1)/2}$ and $supp\,\hat{\phi}_T \subset \{(\xi, \xi^2),\ \xi \in \omega_T\}$.
- $supp\,\hat{\phi}_T \subset \{(\xi, \xi^2),\ \xi\in\omega_T\}$ follows from direct computation ▢

On the orientation of $T_{q,\omega}$ ... (wlog $c_q = 0$)



$c_\omega = (0,0)$ $\quad$ $c_\omega = \binom{\alpha}{0}$, $\alpha > 0$



Where does the point $x^a = \binom{0}{1}$, i.e. $x' = 0$ get mapped to? well, the new x' must simply satisfy $x' + 2c_\omega x_d = 0$, i.e., $x' = -2c_\omega x_d$

$\Rightarrow$ new direction given by $\binom{-2c_\omega x_d}{1}$

$|x' + 2c_\omega x_d| < \sqrt{R} \rightarrow$ if $x' = \binom{\bar{x}}{0}$, $\bar{x} \in (-R^{1/2}, R^{1/2})$ originally and $c_\omega = \binom{\alpha}{0}$, $\alpha > 0$, and $x_d > 0$, then for new tube, $\cancel{x' < \sqrt{R} - 2c_\omega x_d}$ ~~and~~ $\cancel{x' > 2c_\omega x_d - \sqrt{R}}$

~~i.e., $\bar{x} \in [$~~ $\bar{x} + 2\alpha x_d < \sqrt{R}$ and $\bar{x} + 2\alpha x_d > -\sqrt{R}$

$\Rightarrow$ $\bar{x} \in [-\sqrt{R} - 2\alpha x_d, \sqrt{R} - 2\alpha x_d]$, i.e., interval of $\bar{x}$ gets schifted to left

Propagation of singularities $\quad \varphi_{xqw}(\eta) = \eta \, \frac{x' - c_q + 2c_w x_d}{\sqrt{R}} + \eta^2 \frac{x_d}{R}$

$$WF\left(\int d\eta \, \Psi_{q,w}(\eta) e^{2\pi i \varphi_{x,q,w}(\eta)} \right) \subseteq \left\{ (x, \nabla_x \varphi_{x,q,w}(\eta)) : (x,\eta) \in \mathbb{R}^d \times \operatorname{supp} \Psi_{q,w} \text{ s.t. } \nabla_\eta \varphi_{x,q,w}(\eta) = 0 \right\}$$

$$\nabla_\eta \varphi_{xqw}(\eta) = 0 \Leftrightarrow \quad x' - c_q + 2c_w x_d = -2x_d \cdot \eta/\sqrt{R}$$

$$\nabla_x \varphi_{xqw}(\eta) = \begin{pmatrix} \eta/\sqrt{R} \\ \eta/\sqrt{R} \cdot \left(2c_w + \cancel{2}\, \eta/\sqrt{R}\right) \end{pmatrix}$$