6 Connection between restriction and Kakeya's problem

(Mattila – Fourier Analysis & Hausdorff Dimension 2015
– Geometry of sets & measures Sect 4.3
Wolff – Lectures on Harmonic Analysis 2003)

Federer – Sect 2.10
Falconer – Ch I, XIII set

~~6.1 Hausdorff measure~~

~~6.1 restriction → Kakeya maximal conjecture~~

6.1 Hausdorff measures

- By "measure" on a set X we usually mean an outer measure, i.e., a function μ, i.e. $\mu : \{A : A \subseteq X\} \to \mathbb{R}_{\geq 0}$ with $\mu(\emptyset)=0$ which is monotone and countably subadditive, i.e. $\mu\left(\dot{\bigcup}_{n\geq 0} E_n\right) \leq \sum_{n\geq 0} \mu(E_n)$ for any countable, disjoint family $\{E_n\}_{n=1}^{n}$ on which μ is defined
- Borel sets in a metric space X form smallest σ-algebra of subsets of X containing all open (or closed) subsets of X
- Borel measure in X is a measure μ for which Borel sets are measurable and which is regular in sense that $\forall A \subseteq X$ $\exists$ Borel set B such that $A \subseteq B$ and $\mu(A)=\mu(B)$
- Radon measure := locally finite Borel measure, i.e., whenever $\mu(K)<\infty$, K compact

~~• restriction~~

- Sperical Hausdorff measures

Fix $\alpha>0$, and let $E \subseteq \mathbb{R}^d$. For $\epsilon>0$, define $H_\alpha^\epsilon(E) := \inf\left(\sum_{j\geq 1} r_j^\alpha\right)$ where infimum is taken over all countable coverings of E by discs $B_{x_j}(r_j)$ with $r_j<\epsilon$

Clearly: $H_\alpha^\epsilon(E)$ increases as ϵ decreases

Define $\quad H_\alpha(E) := \lim_{\epsilon\to 0} H_\alpha^\epsilon(E)$

Clearly: $\quad H_\alpha^\epsilon(E) \leq H_\beta^\epsilon(E)$ if $\alpha>\beta$ and $\epsilon \leq 1$

$\Rightarrow H_\alpha(E)$ is non-increasing in α

Remarks 6.1 1) If $H_\alpha^1(E)=0$, then $H_\alpha(E)=0$. (For, a covering with $H_\alpha^1(E)<\delta$ necessarily consists of balls with radius $<\delta^{1/\alpha}$ by pigeonhole principle)

2) $H_\alpha(E)=0$ whenever $\alpha>d$ (see Lemma below + next remark)

3) $H_d(E) = \dfrac{1}{Leb(B_0(1))} Leb(E)$ (Federer Thm 2.10.35 with notation given in 2.10.2)

<u>Lemma 6.2</u> ~~Rat~~ Let $E \subseteq \mathbb{R}^d$, then there is a unique number α_0, called Hausdorff dimension of E, $\dim_H E$ s.t. ~~lim~~ $H_\alpha(E) = \infty$ if $\alpha < \alpha_0$ and $H_\alpha(E) = 0$ if $\alpha > \alpha_0$

Pf Define $\alpha_0 := \sup\{\alpha \geq 0 : H_\alpha(E) = \infty\} \rightarrow H_\alpha(E) = 0 \;\forall\, \alpha < \alpha_0$ (since $\alpha \mapsto H_\alpha(E)$ nonincreasing)

For $\alpha > \alpha_0$ let $\beta \in (\alpha_0, \alpha)$ and $M := 1 + H_\beta(E) < \infty \rightarrow$ ~~for~~ $\epsilon > 0$ we have a covering $\bigcup B_{x_j}(r_j) \supseteq E$, $r_j < \epsilon$ with $\sum r_j^\beta \leq M$

$$\Rightarrow \sum r_j^{\alpha - \beta + \beta} \leq \epsilon^{\alpha-\beta} \sum r_j^\beta \leq M \epsilon^{\alpha - \beta} \xrightarrow{\epsilon \to 0} 0 \Rightarrow H_\alpha(E) = 0 \;\forall\, \alpha > \alpha_0 \quad \boxed{E}$$

<u>Remarks 6.3</u> 1) Lemma 6.2 $\Leftrightarrow$ for $0 \leq \beta < \alpha$ and $E \subseteq \mathbb{R}^d$ we have

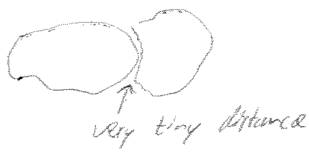
$$H_\beta(E) < \infty \Rightarrow H_\alpha(E) = 0$$
$$H_\alpha(E) > 0 \Rightarrow H_\beta(E) = \infty$$

2) H_α is countably additive on Borel sets $H_\alpha(\dot{\bigcup} E_j) = \sum H_\alpha(E_j)$ and in particular H_α defines a Borel measure

Note that σ-additivity generally fails for H_α^ϵ: ~~example H_α^1(~~

example $H_\alpha(E) + H_\alpha(F) = H_\alpha(E \cup F)$, but

$H_\alpha^1(E \cup F) \neq H_\alpha^1(E) + H_\alpha^1(F)$ for compact disjoint $E, F \subseteq \mathbb{R}^d$



3) H_α not σ-finite if $\alpha < d$ since already $H_\alpha(E) = \infty$ for all ~~A~~ $E \subseteq \mathbb{R}^d$ with $Leb(E) > 0$

4) Standard example of a set $C \subseteq \mathbb{R}$ with $\dim C \in (0,1)$ is Cantor's set $\dim C_\gamma = \frac{\log 2}{\log 1/\gamma}$, $\gamma \in (0, \frac{1}{2})$ (start with $[0,1]$ and remove middle segment of length $1-2\gamma$)

(classic: $\gamma = \frac{1}{3}$)

Although estimating $\dim E$ from above is easy (by just estimating convenient coverings by variational principle), ~~often~~ lower bounds are much harder to obtain as all coverings must be evaluated.

The following lemma due to Frostman (cf. Wolff Prop 8.2 or Mattila 2015, Thm 2.7) transforms the problem of finding measures with good upper bounds ~~to~~ for measures with balls

We state Frostman's lemma only for compact sets. For Borel or more general (3 Suslin (or analytic) sets, see eg Bishop and Peres (2016) or Carleson 1967)

If $E \subseteq \mathbb{R}^d$ is compact, we denote by $P(E)$ the set of all probability measures supported on E.

Proposition 6.4 (Frostman's lemma) Let $0 \leq \alpha \leq d$ and $E \subseteq \mathbb{R}^d$ compact.

Then: $H_\alpha(E) > 0 \Leftrightarrow \exists\, \mu \in P(E)$ s.t. $\mu(B_x(r)) \leq c r^\alpha$ for all $x \in \mathbb{R}^d$, $r > 0$ and a suitable constant $c > 0$

(Equivalently: $\dim(E) = \sup\{0 \leq \alpha \leq d : \exists\, \mu \in P(E) : \mu(B_x(r)) \lesssim r^\alpha \ \forall x \in \mathbb{R}^d, r > 0\}$

"Proof" "$\Leftarrow$" Let $\{B_{x_j}(r_j)\}$ be any covering of E, then

$$1 = \mu(E) \leq \sum \mu_j(B_{x_j}(r_j)) \lesssim c \sum r_j^\alpha$$

which shows $H_\alpha(E) \geq \frac{1}{c}$.

"$\Rightarrow$" The construction of a suitable measure is most easily done using dyadic cubes.

$\mathcal{Q}_k = \{2^{-k} \cdot n + [0, 2^{-k}]^d\}_{n \in \mathbb{Z}^d}$ mesh of dyadic cubes of side length 2^{-k} with vertices at $2^{-k}\mathbb{Z}^d$

Recall the nesting properties: If $Q \in \mathcal{Q}_k$, then there is a unique $\tilde{Q} \in \mathcal{Q}_{k-1}$ such that $Q \subseteq \tilde{Q}$. Furthermore, if we fix $Q_1 \in \mathcal{Q}_{k-1}$, then Q_1 is the union of those $Q \in \mathcal{Q}_k$ with $\tilde{Q} = Q_1$ and the union is disjoint except for edges and vertices.

For a dyadic cube Q, there is $B_x(r) \supseteq Q$ and $r \leq C \cdot \ell(Q)$ length of edge of Q

Likewise, if one fixes $B_x(r)$, then there is a bounded number of dyadic cubes $Q_1, \dots, Q_c$ with $\ell(Q_j) \leq Cr$ and $\bigcup_{j=1}^{c} Q_j \supseteq B_x(r)$

$\Rightarrow$ definition of Hausdorff measure and also the statement "$\mu(B_x(r)) \leq cr^\alpha$" in this proposition, could equally well be given in terms of dyadic cubes.

$\Rightarrow$ except for the values of involved constants:

$$\underset{\text{suitable } c}{\mu(B_x(r)) \leq cr^\alpha \ \forall x \in \mathbb{R}^d, r > 0} \Leftrightarrow \mu(Q) \leq C\, \ell(Q)^\alpha \text{ for all dyadic cubes}$$

Furthermore, if we define

$$h_\alpha^\epsilon(E) := \inf\left(\sum_{Q \in \mathcal{F}} \ell(Q)^\alpha : E \subseteq \bigcup_{Q \in \mathcal{F}} Q\right)$$

where $\mathcal{F}$ runs over all coverings of E by dyadic cubes of side length $\ell(Q) < \epsilon$ and $\quad h_\alpha(E) := \lim_{\epsilon \to 0} h_\alpha^\epsilon(E)$, then we have $C^{-1} H_\alpha^\epsilon(E) \le h_\alpha^\epsilon(E) \le C H_\alpha^\epsilon(E)$ and therefore

$$h_\alpha(E) > 0 \Leftrightarrow H_\alpha(E) > 0$$

Now we return to the actual proof of Frostman.

Wlog assume $E \subseteq [0,1]^d$ and $h_\alpha^1(E) > 0$ (recall Remark 6.1, 1))

Task Find ~~pl~~ $\mu \in P(E)$ s.t. $\mu(Q) \le c\, l(Q)^\alpha$ for all dyadic cubes with $l(Q) \le 1$

We make ~~bl~~ a further reduction:

Claim For each fixed $m \in \mathbb{N}$ it suffices to find a positive measure μ satisfying

(1) μ is supported on the union of cubes $Q \in \mathcal{Q}_m$ which intersect E

(2) $\|\mu\| \ge 1/C$ (C is independent of m !!)

(3) $\mu(Q) \le l(Q)^\alpha$ for all dyadic cubes with $l(Q) \ge 2^{-m}$.

Suppose this can be done and call the resulting measure ~~pl~~ μ_m

By (3) we have bound on $\|\mu_m\|$, so that there is a weak-$*$-limit μ of $\mu_m \xrightarrow[m\to\infty]{*} \mu$, which is, by (1), in particular supported on E. Moreover, by (2) and (3) it suffices $\|\mu\| \ge 1/C$ and $\mu(Q) \le l(Q)^\alpha$ $\forall$ dyadic cubes with $l(Q) \ge 0$

$\Rightarrow$ a suitable scalar multiple of μ gives the desired measure

There are multiple ways to construct the measures μ_m satisfying (1)-(3); the issue is that (2) and (3) are competing conditions, i.e., one must find a measure μ with appropriate support and with total mass roughly as large as possible given that (2) holds.

We follow Carleson (1967), Ch. 2 to that end. (Constructive!)

Fix $m \in \mathbb{N}$ and construct a finite sequence of measures $\nu_m, \nu_{m-1}, \dots \nu_0$ in that order; ν_0 will then be the measure we want.

We start by defining ν_m to be the unique measure satisfying

(4) On each $Q \in \mathcal{Q}_m$, ν_m is a scalar multiple of Lebesgue measure

(5) If $Q \in \mathcal{Q}_m$ and $Q \cap E = \emptyset$, then $\nu_m(Q) = 0$

(6) If $Q \in \mathcal{Q}_m$ and $Q \cap E \neq \emptyset$, then $\nu_m(Q) = 2^{-m\alpha}$

If we set $k = m$, then ν_k has the following properties

(A) $\nu_k \ll \mathrm{Leb}$

(B) $\nu_k(Q) \le l(Q)^\alpha$ if $Q \in \mathcal{Q}_j$, $k \le j \le m$

(C) If ~~$Q_1 \in \mathcal{Q}_{k-1}$~~ $Q_1 \in \mathcal{Q}_k$, then there is a covering $\mathcal{F}_{Q_1}$ of $Q_1 \cap E$ by dyadic cubes in Q_1 s.t. $\nu_k(Q_1) \ge \sum_{Q \in \mathcal{F}_{Q_1}} l(Q)^\alpha$

Now let $1 \le h \le m$ and assume we have constructed an absolutely continuous measure ν_h with properties (1), (B) and (C)

We will now construct ν_{h-1} having the same properties where in (B) and (C) h is replaced by $h-1$.

Namely, to define ν_{h-1} it suffices to define $\nu_{h-1}(Y)$ when Y is contained in a cube $Q \in \mathcal{Q}_{h-1}$. Fix $Q \in \mathcal{Q}_{h-1}$ and consider two cases

(i) $\nu_h(Q) \le l(Q)^\alpha$. In this case we let ν_{h-1} aggree with ν_h on subsets of Q

(ii) $\nu_h(Q) > l(Q)^\alpha$. In this case we let ν_{h-1} aggree with $c \cdot \nu_h$ on subsets of Q, where ~~$c := (\nu_h(Q))^{-1}$.~~ $c := \dfrac{2^{-(h-1)\alpha}}{\nu_h(Q)}$.

Notice that $\nu_{h-1}(Y) \le \nu_h(Y)$ for any set Y and furthermore $\nu_{h-1}(Q) \le l(Q)^\alpha$ whenever $Q \in \mathcal{Q}_{h-1}$. These properties and (B) ~~for~~ for ν_h imply (B) for ν_{h-1}. Moreover (1) for ν_{h-1} follows trivially from (1) for ν_h. To see that ν_{h-1} also satisfies (C), fix $Q \in \mathcal{Q}_{h-1}$. If Q is as in case (ii) then $\nu_{h-1}(Q) = l(Q)^\alpha$, so we can use the covering by the singleton $\{Q\}$. If Q is as in case (i), then for each of the cubes $Q_j \in \mathcal{Q}_h$ whose union is Q, we have the covering of $Q_j \cap E$ associated with (B)? but for ν_h. But since ν_h and ν_{h-1} aggree with each other on subsets of Q, we can simply put these coverings together to obtain a suitable covering of $Q \cap E$. This shows (B) for ν_{h-1} and concludes the inductive step ~~for~~ $\nu_h \mapsto \nu_{h-1}$.

$\Rightarrow$ We have therefore constructed ν_0. It has properties (1) and (3) (since for ν_0 this is equivalent to ~~(A)~~ ~~(A)~~ (B)), and by (C) and the definition of h_α' it has property (2) $\square$

We will now transform Frostman's lemma ($\dim E = \sup_{\alpha \ge 0} \{0 \le \alpha \le d : \exists \mu \in \mathcal{P}(E)$ s.t. $\mu(B_x(r)) \lesssim r^\alpha\}$)

into an integral condition related to Riesz transforms ("Coulombic energies")

~~Unfortunately, the stronger assert~~

α-dimensional (Coulombic) energy of a positive measure μ

$$I_\alpha(\mu) := \int \frac{d\mu(x)\,d\mu(y)}{|x-y|^\alpha}, \quad 0<\alpha<d$$

← Riesz kernel

For simplicity, suppose μ is compactly supported (not needed though in the following)

In this case, $I_{\alpha}(\mu)<\infty \Rightarrow I_\beta(\mu)<\infty$, whenever $\alpha>\beta\geq 0$ (see Lemma below for details)

Define the "mean-field potential"

$$V_\mu^\alpha(x) := \int \frac{d\mu(y)}{|x-y|^\alpha}, \text{ i.e., } I_\alpha(\mu) = \int V_\mu^\alpha(x)\,d\mu(x).$$

(Note that $\Delta V_\mu^\alpha(x)\big|_{\alpha=d-2} = 0$ for $x\notin \mathrm{supp}(\mu)$ since $\Delta \frac{1}{|\cdot|^{d-2}} = 0$ away from origin

Note also $V_\mu^\alpha(x) = \alpha\int_0^\infty \frac{\mu(B_x(r))}{r^\alpha}\frac{dr}{r}$ since $\alpha\int_0^\infty \frac{dr}{r^{\alpha+1}}\int_{|x-y|<r} d\mu(y) = \alpha\int d\mu(y)\int_{|x-y|}^\infty \frac{dr}{r^{\alpha+1}} = \int\frac{d\mu(y)}{|x-y|^\alpha}$)

Unfortunately the stronger equivalence $I_\alpha(\mu)<\infty \Leftrightarrow \exists \mu\in P(E): \mu(B(r))\leq r^\alpha \ \forall E$ is false!

((compared to $\dim E = \sup\{0\leq\alpha\leq d: \exists\mu\in P(E): I_\alpha(\mu)<\infty\}$
$\Downarrow$
$\sup\{\alpha\in[0,d]: \exists\mu\in P(E): \mu(B(r))\lesssim r^\alpha\ \forall r\}$ (by Frostman))

<u>Lemma 6.5</u> (1) If μ is a compactly supported probability measure satisfying $\mu(B(r))\lesssim r^\alpha$, then $I_\beta(\mu)<\infty \ \forall\ \beta<\alpha$

(2) If μ is a compactly supported probability measure satisfying $I_\alpha(\mu)<\infty$, then there is another probability measure ν s.t. $\nu(X)\leq 2\mu(X)$ for all sets X and such that $\nu(B(r))\lesssim r^\alpha$

<u>Proof</u> (1) Suppose wlog $\mathrm{diam}(\mathrm{supp}\,\mu)\leq 1$, then $V_\mu^\beta(x)\lesssim \sum_{j\geq 0} 2^{j\beta}\mu(B_x(2^{-j}))$

Accordingly, if $\mu(B(r))\lesssim r^\alpha$ and $\beta<\alpha$, then $V_\mu^\beta(x)\lesssim\sum_{j\geq 0}2^{j(\beta-\alpha)}\lesssim_{\alpha,\beta} 1$

$\Rightarrow \int V_\mu^\beta(x)\,d\mu(x) = I_\beta(\mu)\lesssim_{\alpha,\beta} 1$

(2) Let $F := \{x\in\mathbb{R}^d: V_\mu^\alpha(x)\leq 2I_\alpha(\mu)\}$. Then, $\mu(F) = \int \mathbb{1}_{\{V_\mu^\alpha(x)\leq 2I_\alpha(\mu)\}}\,d\mu(x)$

$$= \frac{1}{2I_\alpha(\mu)}\int 2I_\alpha(\mu)\mathbb{1}_{\{\dots\}}\,d\mu(x)$$

$$\geq \frac{1}{2I_\alpha(\mu)}\int V_\mu^\alpha(x)\,d\mu(x) = \frac{I_\alpha(\mu)}{2I_\alpha(\mu)} = \frac{1}{2}$$

Let ν be defined by $\nu(X) = \frac{\mu(X\cap F)}{\mu(F)}$. Then $\nu(X)\leq 2\mu(X\cap F)\leq 2\mu(X)$,

so we are left with showing $\nu(B_x(r))\lesssim r^\alpha$, $\forall x\in\mathbb{R}^d$

~~Supp~~ Suppose first $x \in F$. Then $\quad \nu \le 2\mu \quad x \in F \Rightarrow V_\mu^\alpha(x) \le 2 I_\alpha(\mu)$

$$\nu(B_x(r)) = \int_{B_x(r)} \frac{d\nu(y)}{|x-y|^\alpha} |x-y|^\alpha \le r^\alpha V_\nu^\alpha(x) \le 2r^\alpha V_\mu^\alpha(x) \le 4r^\alpha I_\alpha(\mu)$$

Now assume $x \notin F$ and distinguish between the following two cases.

Recall $\nu(X) = \frac{\mu(X \cap F)}{\mu(F)}$, so if r is so small that $F \cap B_x(r) = \emptyset$, then $\nu(B_x(r)) = 0$ trivially; if $B_x(r) \cap F \neq \emptyset$, let $y \in B_x(r) \cap F$. ~~Then~~.

Then $\nu(B_x(r)) = \int d\nu(z)\, \mathbb{1}_{B_x(r)\cap F}(z) \le \nu(B_y(2r)) \lesssim r^\alpha$

contains the above y's; make set bigger by also incorporating balls around these y's

$y \in B_x(r) \cap F$

$y \in F$, can use above result.

$\square$

<u>Proposition 6.6</u> Let $E \subseteq \mathbb{R}^d$ compact. Then $\dim(E) = \sup\{0 \le \alpha \le d : \exists \mu \in P(E) \text{ s.t. } I_\alpha(\mu) < \infty\}$

Frostman: $\exists \mu \in P(E)$ st $\mu(B_x(r)) \lesssim r^\alpha$)

<u>Proof</u> Let $s := \sup\{0 \le \alpha \le d : \exists \mu \in P(E) \text{ st } I_\alpha(\mu) < \infty\}$

If $\beta < s$, then by (2) of Lemma 6.5, E supports a measure with $\mu(B_x(r)) \lesssim r^\beta$

$\Rightarrow$ by Frostman (Prop 6.4) we have $H_\beta(E) > 0$, i.e., $\beta \le \dim E \Rightarrow s \le \dim E$

On the other hand, if $\beta < \dim E$, then, again by Frostman, E supports a measure with $\mu(B_x(r)) \lesssim r^{\beta+\epsilon}$ for sufficiently small $\epsilon \Rightarrow$ by Lemma 6.5 (1), we have $I_\beta(\mu) < \infty$, i.e., $\beta \le s \Rightarrow \dim E \le s$ $\square$

Observe that $I_\alpha(\mu) = \int \frac{d\mu(x)\,d\mu(y)}{|x-y|^\alpha} = \langle \mu, |\cdot|^{-\alpha} * \mu \rangle_{\mathcal{S},\mathcal{S}'} = c_{d,\alpha} \langle \hat\mu, |\cdot|^{-d+\alpha} \hat\mu \rangle$

<u>Corollary 6.7</u> Suppose μ is a compactly supported probability measure on $\mathbb{R}^d$ with

$|\hat\mu(\xi)| \lesssim |\xi|^{-\beta}$ for some $0 < \beta < d/2$, or, more generally

in L^2-average $\int_{B_0(N)} |\hat\mu(\xi)|^2\, d\xi \lesssim N^{d-2\beta}$

$\Rightarrow \dim(\operatorname{supp}(\mu)) \geq 2\beta$

<u>Remark</u> Recall that FT of surface measure of curved ~~surfaces~~ hypersurfaces obeys $\beta = \frac{d-1}{2}$

Pf $\dim(\mathrm{supp}(E)) \underset{\text{Prop 6.6}}{=} \sup\{0 \le \alpha \le d : \exists \mu \in P(E) : I_\alpha(\mu) < \infty\}$

So compute $I_\alpha(\mu)$ for $\alpha < 2\beta$:

$$I_\alpha(\mu) = \int |\hat\mu(\xi)|^2 |\xi|^{-d+\alpha} (1_{|\xi|\le 1} + 1_{|\xi|>1})\, d\xi \lesssim \|\mu\| + \sum_{j\ge 0} 2^{-j(d-\alpha)} \int_{2^j \le |\xi| \le 2^{j+1}} |\hat\mu(\xi)|^2 d\xi$$

$|\hat\mu(\xi)| \le \|\mu\|$

$\int_{|\xi|\le 1} |\xi|^{-d+\alpha} = C_{d,\alpha}$

$$\lesssim_\mu \|\mu\|^2 + \sum 2^{-j(d-\alpha)} \cdot 2^{j(d-2\beta)} \lesssim 1 + \|\mu\| \quad \text{if } 2\beta > \alpha \quad \square$$

Natural question: Does a compact set with dimension α necessarily support a measure μ that satisfies $|\hat\mu(\xi)| \lesssim_\epsilon (1+|\xi|)^{-\alpha/2+\epsilon}$?

No, if one insists on pointwise decay, whereas yes, if L^2-averaged decay, bc. the calculation in the proof above is reversible

In fact there are many sets with positive dimension that do not support any measure whose FT decays at infinity

Example: Consider line segment $E = [0,1] \times \{0\} \subseteq \mathbb{R}^2$ with $\dim E = 1$; but if μ is any measure on E, then $\hat\mu(\xi) = \hat\mu(\xi_1)$ only depends on ξ_1, so in particular it doesn't decay along ξ_2-direction at all (unless $\hat\mu = 0$)

More dramatically, it's actually hard to show that a set E with given dimension α supports a measure s.t. $|\hat\mu(\xi)| \lesssim_\epsilon (1+|\xi|)^{-\alpha/2+\epsilon}$. ($\to$ Kaufmann see, eg. Wolff, Ch 9)

6.2 ~~Definitions~~ Kakeya conjectures

A Besicovitch, or Kakeya set is a compact set $E \subseteq \mathbb{R}^d$ which contains a unit line segment in every direction, i.e.,

$$\forall\, \omega \in S^{d-1}\ \exists\, x \in \mathbb{R}^d : x + t\omega \in E \quad \forall\, t \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

Besicovitch showed that such sets may be extremely small (→ Fefferman's YT video)

<u>Thm 6.8 (Besicovitch)</u> If $d \geq 2$, then there are Kakeya sets with Lebesgue measure zero.

But how small can they actually be?
There are several ways one can ask this question mathematically precise.

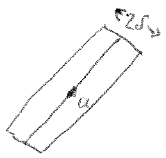
<u>Kakeya set conjecture</u> Besicovitch sets in $\mathbb{R}^d$ necessarily have Hausdorff dimension d.

This is open in $d \geq 3$ but was initially solved by R.O. Davies 1971 in $d=2$

There is also a more quantitative formulation of this problem in terms of a maximal function

For any $0 < \delta \ll 1$ and $\omega \in S^{d-1}$, and $a \in \mathbb{R}^d$, introduce the δ-box around the unit line segment in direction ω, centered at a as follows

$$T_\omega^\delta(a) := \{ x \in \mathbb{R}^d : |(x-a)\cdot\omega| \leq \tfrac{1}{2},\ |\underbrace{(x-a)^\perp}_{= (x-a) - \langle (x-a), \omega \rangle \omega}| \leq \delta \}$$



Kakeya maximal fct for $f \in L^1_{loc}(\mathbb{R}^d)$ defined by

$$M_\delta : L^1_{loc}(\mathbb{R}^d) \to L^1_{loc}(S^{d-1})$$

$$f \mapsto (M_\delta f)(\omega) := f_\delta^*(\omega) := \sup_{a \in \mathbb{R}^d} \frac{1}{|T_\omega^\delta(a)|} \int_{T_\omega^\delta(a)} |f|$$

<u>Goal</u> $\forall\, \epsilon > 0\ \exists\, C_\epsilon > 0$ s.t. $\|f_\delta^*\|_{L^p(S^{d-1})} \leq C_\epsilon \delta^{-\epsilon} \|f\|_{L^p(\mathbb{R}^d)}$, i.e, "$\epsilon$-losses" are admissible.

<u>Remarks 6.9</u>

Remarks 6.9 (1) By definition, one trivially has

$$\|f_\delta^*\|_\infty \le \|f\|_\infty \quad \forall \delta$$

$$\|f_\delta^*\|_\infty \le \delta^{-(d-1)}\|f\|_1$$

(2) If $d \geq 2$ and $p < \infty$, then there can be no bound of the form

$$\|f_\delta^*\|_q \le c\|f\|_p \quad \text{for } \delta\text{-independent } c.$$

To that end, consider a zero-measure Kakeya set E, and let E_δ denote its δ-neighborhood, and let $f = \mathbb{1}_{E_\delta}$

$\Rightarrow$ $f_\delta^*(\omega) = 1$ for all $\omega \in S^{d-1}$, so $\|f_\delta^*\|_q \sim 1$. But on the other hand $\|f\|_p^p = \int_{E_\delta} dx = |E_\delta| \xrightarrow{\delta \to 0} 0$ for any $p < \infty$.

(3) The claimed $\|f_\delta^*\|_p \lesssim_\epsilon \delta^{-\epsilon}\|f\|_p$ cannot hold if ~~…~~ $p < d$.

To that end take $f = \mathbb{1}_{B_0(\delta)}$ $\Rightarrow$ $T_\omega^\delta(0) \supseteq B_0(\delta)$ $\forall \omega$, so

$$f_\delta^*(\omega) = \frac{|B_0(\delta)|}{|T_\omega^\delta(0)|} \sim \delta \Rightarrow \|f_\delta^*\|_p \sim \delta$$

However $\|f\|_p \sim \delta^{d/p}$ which goes faster to zero than δ, if $p < d$.

Kakeya maximal conjecture $\forall \epsilon > 0 \; \exists C_\epsilon > 0$ s.t. $\|f_\delta^*\|_{L^p(S^{d-1})} \le C_\epsilon \delta^{-\epsilon}\|f\|_{L^p(\mathbb{R}^d)}$,

whenever $p \in [d, \infty]$.

(Clearly if ~~suffic~~ one proves the estimate for $p = d$, then it follows for all other p by interpolation)

This was proved in $d=2$ by Cordoba using a nice geometric argument, but is open in $d \geq 3$

This type of result is "easy" bc. Hilbert space techniques are available

Proposition 6.10 (Kakeya maximal $\Rightarrow$ Kakeya set conjecture)

If $\|f_\delta^*\|_p \lessapprox_\epsilon \delta^{-\epsilon}\|f\|_p$ holds for some $p<\infty$, then Besicovitch sets in $\mathbb{R}^d$ necessarily have Hausdorff dimension $=d$.

Proof: Let E be a Besicovitch set and $\{B_j\}_j = \{B(x_j,r_j)\}_j$ a covering with all $r_j \le \frac{1}{100}$

Let $\mathcal{J}_k = \{j : 2^{-k} \le r_j \le 2^{-(k-1)}\}$

For every $\omega \in S^{d-1}$, E contains a unit line segment I_ω parallel to ω

Let $S_k = \{\omega \in S^{d-1} : |I_\omega \cap \bigcup_{j\in\mathcal{J}_k} B_j| \ge \frac{1}{100k^2}\}$, then $\bigcup_{k=1}^{\infty} S_k = S^{d-1}$, since

$\sum_k |I_\omega \cap \bigcup_{j\in\mathcal{J}_k} B_j| \ge |I_\omega| = 1$ and $\sum \frac{1}{100k^2} < 1$.

Let $f = \mathbb{1}_{F_k}$, where $F_k = \bigcup_{j\in\mathcal{J}_k} B_{x_j}(10r_j)$

For $\omega \in S_k$, we have $|T_\omega^{2^{-k}}(\underset{\text{midpoint of } I_\omega}{a_\omega}) \cap F_k| \overset{\text{Def of } S_k}{\gtrsim} \frac{1}{100k^2} \cdot |T_\omega^{2^{-k}}(a_\omega)|$, $\omega \in S_k$

$\Rightarrow f^*_{2^{-k}}(\omega) = \frac{1}{|T_\omega^{2^{-k}}|} \sup_a \int_{T_\omega^{2^{-k}}(a)} \mathbb{1}_{F_k} \ge \frac{|F_k \cap T_\omega^{2^{-k}}(a_\omega)|}{|T_\omega^{2^{-k}}|} \gtrsim \frac{1}{100k^2}$ if $\omega \in S_k$

$\Rightarrow \|f^*_{2^{-k}}\|_p \ge \|f^*_{2^{-k}}\|_{L^p(S_k)} \gtrsim \frac{\sigma_{S^{d-1}}(S_k)^{1/p}}{100k^2}$ $(*)$

On the other hand, the maximal conjecture implies

$\|f^*_{2^{-k}}\|_p \le C_\epsilon 2^{k\epsilon} \|f\|_p \le C_\epsilon 2^{k\epsilon} (|\mathcal{J}_k| 2^{-(k-1)d})^{1/p} \lesssim_{d,\epsilon} 2^{k\epsilon}(|\mathcal{J}_k| 2^{-kd})^{1/p}$

$\Rightarrow$ Combining this with the upper bound $(*)$ shows

$\sigma_{S^{d-1}}(S_k) \lessapprox_\epsilon \cancel{2^{k(p\epsilon-d)}}\ 2^{k(p\epsilon-d)} k^{2p} |\mathcal{J}_k| \lesssim 2^{k(2p\epsilon-d)} |\mathcal{J}_k|$

$\Rightarrow \sum_{j\ge 0} r_j^{d-2p\epsilon} = \sum_k \sum_{j\in\mathcal{J}_k} r_j^{d-2p\epsilon} \gtrsim \sum_k 2^{-k(d-2p\epsilon)} |\mathcal{J}_k| \gtrsim \sum_k \sigma_{S^{d-1}}(S_k)$

$\ge \sigma(\bigcup S_k) = 1$

$\Rightarrow \sum r_j^{d-2p\epsilon} \gtrsim 1$ whenever ~~$d-2p\epsilon$~~ $p\in[d,\infty)$, i.e., $\sum r_j^\alpha \gtrsim 1$ $\forall\ \alpha<d$ $\Rightarrow \dim_H E \ge d$

$\Rightarrow \dim_H(E) = d$ (since $\dim_H(E) \le d$ trivially) ∎

(A slight generalization of this argument can be found in Sogge's book, Prop 9.1.5, see also Lemma 9.1.3)

~~We wi~~

6.3 Restriction conjecture $\Rightarrow$ Kakeya conjecture

To see this, we use the following duality argument

Lemma 6.11 Let $1<p<\infty$ be such that it has the following property:
If $\{w_k\} \subseteq S^{d-1}$ is a maximal δ-separated set (in $\mathbb{R}^d$) and if $\delta^{d-1}\sum_k y_k^{p'} \le 1$, then for any choice of points $a_k \in \mathbb{R}^d$, we have $\| \sum_k y_k \mathbb{1}_{T_{w_k}^{\delta}(a_k)} \|_{p'} \le A$.
Then, there is a bound $\| f_\delta^* \|_{L^p(S^{d-1})} \lesssim A \|f\|_p$

For a metric set (X,d), a subset $A \subseteq X$ is said to be maximal δ-separated if
- $\forall\, x \neq y$ in A, we have $d(x,y) \geq \epsilon$
- ($\forall\, z \in X \setminus A\ \exists\, w \in A\ d(z,w) < \epsilon$)

$\to$ here: δ-separated subset of S^{d-1} has cardinality $\lesssim \delta^{-(d-1)}$

Pf Suppose $\{w_k\}_k$ is a maximal δ-separated subset of S^{d-1}.
Observe that if $|w-w'|<\delta$, then $f_\delta^*(w) = \sup_a \frac{1}{|T_w^\delta(a)|} \int_{T_w^\delta(a)} |f| \le C f_\delta^*(w')$ since each $T_w^\delta(a)$ can be covered by a bounded number of tubes $T_{w'}^\delta(a')$.

$\Rightarrow$ Using $\|f\|_{\ell^p} = \sup_{g \in \ell^{p'},\ \|g\| \le 1} |\langle g, f\rangle|$ with $\vec{f} = (f_k)_{k \in W}$, $f_k = \delta^{(d-1)/p} |f_\delta^*(w_k)|$ shows
$\vec{g} = (g_k)_{k\in W}$, $g_k = y_k \delta^{(d-1)/p'}$ ↑ satisfies above assumption

$$\|f_\delta^*\|_p \le \Big(\sum_k \int_{B(w_k,\delta)} dw \underbrace{|f_\delta^*(w)|^p}_{\le C^p f_\delta^*(w') \text{ for } |w-w'|<\delta;\ \text{take } w'=w_k} \Big)^{1/p} \le C \Big(\sum_k f_\delta^*(w_k)^p \cdot |B_{w_k}(\delta)| \Big)^{1/p}$$

$$\sim C \|\vec{f}\|_{\ell^p} = C \sum_k \langle \vec{g}, \vec{f} \rangle = C \cdot \delta^{d-1} \sum_k y_k |f_\delta^*(w_k)|$$

$$= C \delta^{d-1} \sum_k y_k \cdot \underbrace{\frac{1}{|T_{w_k}^\delta(a_k)|}}_{\sim \delta^{1-d}} \int_{T_{w_k}^\delta(a_k)} |f| \qquad \text{for some suitable } \{a_k\}$$

$$\sim C \int \Big(\sum_k y_k \mathbb{1}_{T_{w_k}^\delta(a_k)} \Big) |f| \underset{\text{Hölder}}{\le} C \|f\|_p \underbrace{\Big\| \sum y_k \mathbb{1}_{T_{w_k}^\delta(a_k)} \Big\|_{p'}}_{\le A \text{ by assumption}} \le C \cdot A \|f\|_p$$

$\square$

Thm 6.12 (Restriction $\Rightarrow$ Kakeya)

Suppose $\|\hat{f}\|_{L^q(S^{d-1})} \le \|f\|_{L^p}$ whenever $p < \frac{2d}{d+1}$ & $\frac{d+1}{p'} \le \frac{d-1}{q}$. Then $\|f_\delta^*\|_d \lesssim_\epsilon \delta^{-\frac{d}{\epsilon}}\|f\|_d$

~~then for a maximal δ-separated set $\Omega \subseteq S^{d-1}$ and a collection $\{T_\omega^\delta(0)\}_{\omega\in\Omega}$, we have $\|\sum_{\omega\in\Omega} 1\!\!1_{T_\omega^\delta(0)}\|_{L^{d/(d-1)}(\mathbb{R}^d)} \lesssim_\epsilon \delta^{-\epsilon}\left(\sum_{\omega\in\Omega}|T_\omega^\delta|\right)^{\frac{d-1}{d}}$~~

~~note $\left(\frac{d}{d-1}\right)' = d$~~

Pf. By restriction, we know that $\|(g\,d\sigma)^\vee\|_{L^{2d/(d-1)}(\mathbb{R}^d)} \le \|g\|_{L^{2d/(d-1)}(S)}$ barely fails, (p')

all we need here $\rightarrow$ $\boxed{\|(g\,d\sigma)^\vee\|_p \le \|g\|_p, \quad p > \frac{2d}{d-1}}$ (recall exercise 4.4 on equivalent formulations of restriction

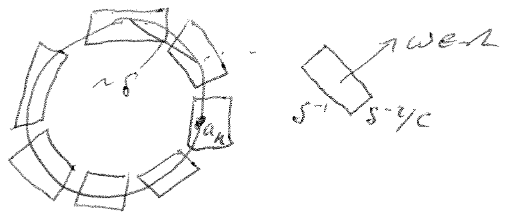
but we do have the estimates $\|(g\,d\sigma)^\vee\|_{L^{2d/(d-1)}(B_{x_0}(\delta^{-1}))} \lesssim_\epsilon \delta^{-\epsilon}\|g\|_{L^{2d/(d-1)}(S^{d-1})}$

via restriction + the ϵ-removal lemma ~~which states "$R_S(p\to p;\alpha) \Rightarrow R_S(q\to q)$ whenever $\frac{1}{q} > \frac{1}{p} + \frac{1}{\log 1/\alpha}$~~ $(p<2)$ $0<\alpha\ll 1$

$\|\hat{f}\|_{L^p(S)} \le \delta^{-\alpha}\|f\|_{L^p(B_{x_0}(\delta^{-1}))}$

Tao $\downarrow$ sort later

Let $g = \sum_\kappa 1\!\!1_\kappa \cdot y_\kappa$ where κ are the caps associated to the $\delta^{+1}\times\ldots\times\delta^{+1}\times\frac{1}{C}\delta^{+2}$ boxes with $C>1$ so large that these boxes do not overlap if we arrange them along the δ-seperated subset of directions



$\kappa_\omega = \tilde{T}_\omega^\delta(a_\omega) \cap S^{d-1}$

T_ω^δ dimensions $\delta^{+1}\times\ldots\times\delta^{+1}\times\frac{\delta^{+2}}{C}$

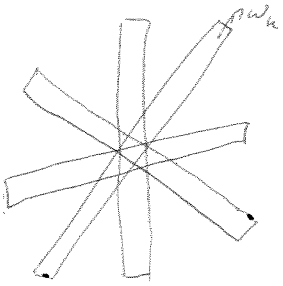
$\rightarrow$ all of the caps are Knapp examples

Now let's randomize these Knapp examples to exploit cancellation effects which reflect that the dual tubes are heavily compressed in position space.

$\rightsquigarrow g = \sum_\kappa \varepsilon_\kappa y_\kappa 1\!\!1_\kappa$; since the caps are disjoint, we have
(ε_κ: random signs)

$$\|g\|_{L^q(S^{d-1})}^q \sim \sum_\kappa \int_\kappa |y_\kappa|^q \sim \sum_\kappa |y_\kappa|^q \cdot \delta^{-(d-1)}$$

On the other hand $\widehat{g\,d\sigma}(x) = \sum \varepsilon_\kappa y_\kappa \cdot \delta^{d-1}\chi_{\tau_\kappa}$ where χ_{τ_κ} are Schwartz-fcts adapted to the dual tubes τ_κ oriented along ω_κ, of dimensions $\delta^{-1}\times\ldots C\delta^{-2}$ centered at origin, (at least if the ε_κ are real-valued...)



By the randomization, we can use Khintchine's inequality

$$\Big(\sum |a_n|^2\Big)^{p/2} \sim \cancel{\mathbb{E}_\eta \Big|\sum \eta_n a_n\Big|^p} \quad \mathbb{E}_\eta \Big|\sum \eta_n a_n\Big|^p$$

$$\Rightarrow \mathbb{E}_\eta \|(g\,d\sigma)^\vee\|_q^q \sim \mathbb{E}\int \Big|\sum \eta_n y_n \delta^{d-1} \chi_{\tau_n}\Big|^q$$

$$\sim \int \Big(\sum |y_n|^2 \delta^{2(d-1)} \chi_{\tau_n}(x)^2|\Big)^{q/2}$$

$$\gtrsim \delta^{q(\delta-1)} \int \Big(\sum |y_n|^2 |\tilde{\chi}_{\tau_n}(x)|^2\Big)^{q/2}$$

where $\tilde{\chi}_\tau = \chi_\tau|_\tau$, i.e., we remove all Schwartz tails

$\Rightarrow$ by restriction, we get for any $q > \frac{2d}{d-1}$ that

$$\delta^{q(d-1)} \int \Big(\sum_n |y_n|^2 |\chi_{\tilde{\tau}_n}(x)|^2\Big)^{q/2} \lesssim \sum |y_n|^q \delta^{-(d-1)}$$

Writing $z_n = y_n^2$ and $p' = q/2$, this is equivalent to the statement

"If $\delta^{d-1} \sum_n (z_n)^{p'} \leq 1$ then $\|\sum_n z_n \chi_{\tilde{\tau}_n}\|_{p'} \lesssim \delta^{-2(d-1)}$ for any $p' > \frac{d}{d-1}$"

We now rescale the $\tilde{\tau}_n$ ($\delta^{-1} \times \dots \times \delta^{-1} \times \delta^{-2}$ - tubes) to T_n, i.e. ~~$\delta \times \dots \times \delta$~~ $\delta \times \dots \times \delta \times 1$ - tubes, ($\sim$ rescale $\times\, \delta^2$)

the ones, we dealt with earlier, in particular in Lemma 6.11

$\Rightarrow$ "If $\delta^{d-1} \sum_n (z_n)^{p'} \leq 1$, then $\|\sum_n z_n \tilde{\chi}_{T_n}\|_{p'} \lesssim \delta^{-2(d-1+d/p')}$ $\forall\, p' > \frac{d}{d-1}$"

Observe that $d-1+\frac{d}{p'} \nearrow 0$ as $p' \searrow \frac{d}{d-1}$, so for any $\epsilon > 0$ we showed

"If $\delta^{d-1} \sum (z_n)^{p'} \leq 1$, then $\|\sum_n z_n \tilde{\chi}_{T_n}\|_{p'} \lesssim_\epsilon \delta^{-\epsilon^4}$ if p is close to d"

$\Rightarrow$ By Lemma 6.11 this implies that for any $\epsilon > 0$, we have $\|f_\delta^*\|_p \lesssim_\epsilon \delta^{-\epsilon} \|f\|_p$ provided $p < d$ is close enough to d. Interpolating this with the $\|f_\delta^*\|_\infty \leq \|f\|_\infty$ bound we obtain $\|f_\delta^*\|_d \lesssim_\epsilon \delta^{-\epsilon} \|f\|_d$, as desired $\boxtimes$

reflects extreme compression of τ_n. If $z_n = 1$ then LHS satisfied and RHS reads $\|\sum_n 1\!\!1_{\tau_n}\|_{p'} \lesssim \delta^{-2(d-1)}$

$\frac{(d+1)(d-1)}{d} + d - 1$ would be justified for disjoint tubes

$= 2d - \frac{1}{d}$

triangle ineq would give $\delta^{-(\frac{d+1}{p'} + d - 1)} = \delta^{-(2d-\frac{1}{d})}$ worse by a factor $\delta^{2-1/d}$ $\Leftarrow$ # of tubes $\sim \delta^{-(d-1)}$

each of τ_n has $L^{p'}$-norm $\sim \delta^{-(d+1)/p'}$

6.4 Proof of Kakeya maximal conjecture in $d=2$ (Cordoba 1977, Bourgain 1991)

Thm 6.13 If $d=2$, then $\|f_\delta^*\|_{L^2(S^1)} \lesssim \sqrt{\log 1/\delta}\, \|f\|_{L^2(\mathbb{R}^2)}$

Proof of Thm 6.13 due to Bourgain (using Fourier analysis)

Wlog suppose $f \geq 0$ and let $\rho_\delta^\omega(x) := \frac{1}{2\delta} 1_{T_\omega^\delta(0)}$, then $f_\delta^*(\omega) = \sup_{a \in \mathbb{R}^2} (\rho_\delta^\omega * f)(a)$.

Let $0 \leq \phi \in \mathcal{S}(\mathbb{R})$ with $\phi(x) \geq 1$ for $|x| \leq 1$ and $\hat{\phi}$ compactly supported

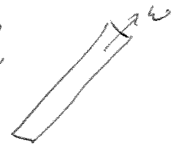
Let $\psi(x) = \phi(x_1) \cdot \frac{1}{\delta} \phi\left(\frac{x_2}{\delta}\right)$ and note $\psi \geq \rho_\delta^\omega$ if $\omega = \hat{e}_1$, i.e. $f_\delta^*(e_1) \leq \sup_a (\psi * f)(a)$

Analogously, $f_\delta^*(\omega) \leq \sup (\psi_\omega * f)(a)$ where $\psi_\omega = \psi \circ R_\omega$ for an appropriate $R_\omega \in SO(2)$

$\Rightarrow f_\delta^*(\omega) \leq \|\psi_\omega * f\|_\infty \leq \|\hat{\psi}_\omega \hat{f}\|_{L^1} = \int |\hat{\psi}_\omega(\xi)| |\hat{f}| \overset{CS}{\leq} \left(\int |\hat{\psi}_\omega| |\hat{f}|^2 \langle \xi \rangle d\xi\right)^{1/2} \left(\int d\xi \frac{|\hat{\psi}_\omega|}{\langle \xi \rangle}\right)^{1/2}$

To compute the second factor we use $\hat{\psi}_\omega = \hat{\psi} \circ R_\omega$ and $\hat{\psi} = \hat{\phi}(\xi_1) \hat{\phi}(\delta \xi_2)$ s.t.

$|\hat{\psi}_\omega| \lesssim 1$ and $\hat{\psi}$ is supported on a rotated $1 \times \delta^{-1}$ ~~rectangle~~ box P_ω



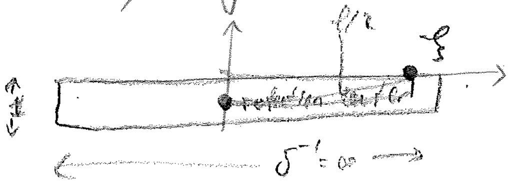
$\Rightarrow \int \frac{|\hat{\psi}_\omega(\xi)|}{\langle \xi \rangle} \lesssim \int_{P_\omega} \frac{d\xi}{\langle \xi \rangle} \sim \int_1^{1/\delta} \frac{d\xi_2}{\xi_2} \sim \log 1/\delta$

$\Rightarrow \|f_\delta^*\|_2^2 \lesssim \log \frac{1}{\delta} \cdot \int_{S^1} \left(\int d\xi \langle \xi \rangle |\hat{f}|^2 |\hat{\psi}_\omega(\xi)| \right) d\omega \qquad (*)$

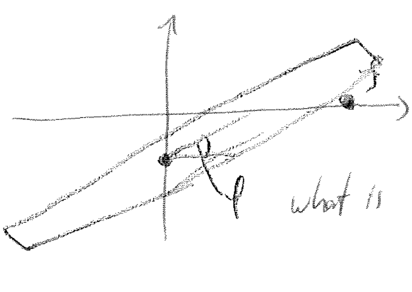
HW **Claim:** For fixed $\xi \in \mathbb{R}^d$, ~~the set~~ one has $|\{\omega \in S^{d-1} : |\hat{\psi}_\omega(\xi)| > 0\}| \lesssim \langle \xi \rangle^{-1} \quad \forall \delta > 0$

Believing the claim shortly shows $(*) \lesssim \log \frac{1}{\delta} \int d\xi |\hat{f}|^2$ □

Proof of claim By symmetry it suffices to consider $d=2$, ~~wlog~~ i.e. $\xi \equiv (\xi_1, 0)$ with $|\xi_1| \geq 1000$, say. (Independently of δ, although we still keep in mind that ψ_ω is $\sim$ constant on the $1 \times \delta^{-1}$ tube. In fact we may even assume that it's infinitely elongated)



now let's rotate and see how $|\{\omega \in S^1 : |\hat{\psi}_\omega(\xi_1, 0)| > 0\}| \leq \int_S 1_{T_\omega}(\xi) \sim \varphi \sim \frac{1}{|\xi|}$ gets smaller



what is φ?

~~tan~~ $\sin \frac{\varphi}{2} = \frac{1/2}{|\xi|} \sim \sin \varphi/2 \sim \varphi/2$

$\Rightarrow \varphi \sim |\xi|^{-1}$

Remark For $d \geq 3$ the $\log 1/\delta$ in $\int \frac{|\hat{\sigma}_\omega(\xi)|}{(\cdot)} \lesssim \log\frac{1}{\delta}$ becomes $\delta^{-(d-2)}$ and hence

$$\| f_\delta^* \|_{L^2(S^{d-1})} \lesssim \delta^{-(d-2)/2} \| f \|_{L^2(\mathbb{R}^d)}$$ which is the best possible L^2 bound

Proof of Thm 6.13 due to Cordoba (geometric)

By lemma 6.11 it suffices to prove that for any sequence $\{y_n\}$ with $\delta \sum y_n^2 = 1$ and any maximal δ-separated subset $\{w_n\} \subseteq S^1$, we have for any choice of $\{a_n\} \subseteq \mathbb{R}^2$,

$$\Big\| \sum_n y_n \mathbb{1}_{T^\delta_{w_n}(a_n)} \Big\|_2 \lesssim \sqrt{\log 1/\delta}$$

HW The relevant geometric fact is $|T^\delta_{w_n}(a) \cap T^\delta_{w_\ell}(b)| \lesssim \frac{\delta^2}{\delta + |w_n - w_\ell|} \sim \min\{\delta, \frac{\delta^2}{|w_n - w_\ell|}\}$

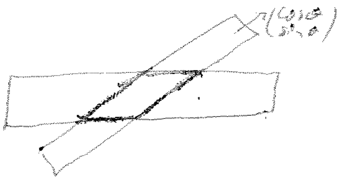
Clearly it suffices to consider $a = b$, $e_\ell = e_1 = (1,0)$, and that the angle θ between $w_n = \binom{\cos\theta}{\sin\theta}$ and e_1 is $\leq \pi/2$. Since $|T_w^\delta(a)| = \delta$, the first bound is clear.

Now suppose $|w_n - e_1| = \sqrt{2}\sqrt{1-\cos\theta} \geq \delta$ (otherwise min will be attained by δ)

$\Leftrightarrow \theta > \sqrt{2}\,\delta$ by using $\cos\theta < 1 - \frac{\theta^2}{4}$

In this case $\sin\theta > \frac{\theta}{2} > \delta/2$ and we have $\frac{\delta}{2\sin\theta} < 1$. Using $\cos\theta \geq 1 - \theta^2$ and the formula for the surface area of a parallelogram, we get

$$|T^\delta_{w_n} \cap T^\delta_{w_\ell}| = 4\,\frac{\delta}{2} \cdot \frac{\delta}{2\sin\theta} \leq \frac{2\delta^2}{\theta} \leq \frac{2\sqrt{2}\,\delta^2}{\sqrt{2}\sqrt{1-\cos\theta}} = \frac{2\sqrt{2}\,\delta^2}{|w_n - w_\ell|} \quad \square$$



Using this geometric observation allows us to estimate

$$\Big\| \sum y_n \mathbb{1}_{T^\delta_{w_n}} \Big\|_2^2 = \sum_{n,\ell} y_n y_\ell \, |T^\delta_{w_n}(a_n) \cap T^\delta_{w_\ell}(a_\ell)| \lesssim \sum_{n,\ell} y_n y_\ell \frac{\delta^2}{\delta + |w_n - w_\ell|}$$

$$= \sum_n \sqrt{\delta}\, y_n \sqrt{\delta}\, y_\ell \underbrace{\frac{\delta}{\delta + |w_n - w_\ell|}}_{= K_{n\ell}} \underset{CS}{\leq} \|\sqrt{\delta}\, y_n\|_{\ell^2_n} \overbrace{\Big\| \sum_\ell K_{n\ell} \sqrt{\delta}\, y_\ell \Big\|_{\ell^2_n}}^{\lesssim \ln\frac{1}{\delta} \|\sqrt{\delta}\, y_\ell\| \text{ by Schur test}} \lesssim \ln\frac{1}{\delta} \underbrace{\|\sqrt{\delta}\, y_n\|_2^2}_{=1}$$

ℓ^2_n means summation over n

Now recall that the set $\{w_n\}$ is δ-separated, i.e., for fixed n there are at most δ^{-1} many summands in the ℓ-summation. Moreover, since the angle between w_n and w_ℓ is given by $\sim \delta|n-\ell|$, ($w_n \cdot w_\ell = \cos\theta$

we have $|w_n - w_\ell| = \sqrt{2}\sqrt{1 - \cos(\delta|n-\ell|)} \geq \sqrt{\frac{2}{100}}\,\delta|n-\ell|$ for $\delta|n-\ell| < 1$

$\Rightarrow \sup_n \sum_\ell \frac{\delta}{\delta + |w_n - w_\ell|} \lesssim \sum_{\ell < 1/\delta} \frac{\delta}{\delta + \delta\ell} \sim \log 1/\delta$ $\rightarrow$ by a symmetric Schur test ($K_{n\ell} = K_{\ell n}$), we are done $\square$

6.5 Can Kakeya help to prove restriction? (Bourgain 1991)

Thm 6.14 Suppose we have an estimate (*) $\| \sum_j \mathbb{1}_{T^\delta_{\omega_j}} \|_{q'} \lesssim_\epsilon \delta^{-(\frac{d}{q}-1+\epsilon)}$ for any given $\epsilon > 0$ and some fixed $q > 2$. Then $\|(f d\sigma)^\vee\|_p \lesssim_p \|f\|_{L^\infty(S^{d-1})}$ for some $p < \frac{2(d+1)}{d-1}$

recall equivalent formulations of restriction (ex 4.4)

Remark (1) The geometrical statement corresponding to (*) is that Besicovitch sets in $\mathbb{R}^d$ have Hausdorff dimension at least q (see, e.g. Sogge 9.1.5)

We will only sketch the proof below for $d=3$ (see Wolff Thm 10.6). Recall that in this case we already know the bounds $\|(f d\sigma)^\vee\|_{L^4} \lesssim \|f\|_{L^2}$ (Tomas-Stein) and

$$\|(f d\sigma)^\vee\|_{L^2(B_0(R))} \lesssim R^{1/2} \|f\|_{L^2} \quad \text{(exercise!)}$$

$\alpha = d-1 = 2$

$\Rightarrow$ interpolation gives $\|(f d\sigma)^\vee\|_{L^p(B_0(R))} \lesssim R^{2/p-1/2} \|f\|_{L^2(S^2)}$, $2 < p < 4$

We now show that the power $\frac{2}{p} - \frac{1}{2}$ of R can be lowered by ϵ, if $L^2(S^2)$ norm on RHS is replaced by L^∞ norm

Prop. 6.15 Let $d=3$, $2<p<4$ and assume $\| \sum \mathbb{1}_{T^\delta_{\omega_j}} \|_{q'} \lesssim \delta^{-(\frac{d}{q}-1+\epsilon)}$ holds for some $q > 2$. Then $\|(f d\sigma)^\vee\|_{L^p(B_0(R))} \lesssim R^{\alpha(p)} \|f\|_{L^\infty(S^{d-1})}$ where $\alpha(p) < \frac{2}{p} - \frac{1}{2}$

$\to$ This implies Thm 6.14 for all p s.t. $\frac{2}{p} - \frac{1}{2} \le 0$, i.e., $p < 4$

Remark Similar type of lemma due to Tao (1999) "ϵ-removal lemma" (see Appendix A in notes)

Heuristic proof of Prop 6.15

By homogeneity we may assume $\|f\|_{L^\infty(S^{d-1})} = 1$ and let $\delta = R^{-1}$ in above assertion. Now cover S^2 by spherical caps $S_j = \{\omega \in S^2 : |1 - \omega \cdot e_j| < \delta\}$ where the $\{e_j\}$ shall form a maximal $\sqrt{\delta}$ separated subset of S^2. Then decompose

$f = \sum f_j$, supp $f_j \subseteq S_j$ and let $G_j = (f_j d\sigma)^\vee$, $G = (f d\sigma)^\vee = \sum G_j$.

By the uncertainty principle, G_j roughly constant on dual tubes of dim $R^{\frac{1}{2}} \times R^{\frac{1}{2}} \times R$ pointing in direction e_j. ~~Th~~ We will ~~not~~ now cheat and neglect their Schwartz tails

Now cover $B_0(R)$ with disjoint cubes Q of side length $\sqrt{R}$. For each Q we denote by $N(Q)$ the number of tubes τ_j which intersect it.

Note that $G|_Q = \sum_j G_j|_Q$ where the sum is only over those j for which $\tau_j \cap Q \neq \emptyset$. Using this and the family of estimates $\|(f d\sigma)^\vee\|_{L^p(B_0(R))} \lesssim R^{\frac{2}{p}-\frac{1}{2}}\|f\|_{L^2}$ obtained from interpolation, we can estimate for $2 \le p \le 4$,

$$\|G\|_{L^p(Q)} \lesssim R^{\frac{1}{2}(\frac{2}{p}-\frac{1}{2})} \Big\|\sum_{j:\tau_j\cap Q\neq\emptyset} f_j\Big\|_{L^2(S^2)} \lesssim R^{\frac{1}{2}(\frac{2}{p}-\frac{1}{2})}\sqrt{N(Q)\cdot|S_\ell|} \quad \text{(any } \ell)$$

$$\underset{R=1/\delta}{\sim} \delta^{\frac{3}{4}-\frac{1}{p}}\sqrt{N(Q)}$$

$\Rightarrow$ Summing over Q yields

$$\|G\|^p_{L^p(B_0(R))} \lesssim \delta^{\frac{3p}{4}-1}\sum_Q N(Q)^{p/2} \sim \delta^{\frac{3p}{4}+\frac{1}{2}}\Big\|\sum_j 1_{\tau_j}\Big\|^{p/2}_{p/2}$$

where we used $\|\sum 1_{\tau_j}\|^{p/2}_{p/2} = \sum N(Q)^{p/2}\cdot|Q| = \delta^{-3/2}\sum N(Q)^{p/2}$

Now let $p = 2q'$ where q' is as in the exponent in assumed bound in Thm 6.14) and assume that p is sufficiently close to 4 (otherwise interpolate between assumpt. of Thm 6.14 and $\|f_\delta^*\|_2 \lesssim \delta^{-(d-2)/2}\|f\|_2$ from Pf of Thm 6.13 due to Bourgain for $d \ge 3$) to get p close to 4.

By ~~$\|\sum 1_{T_{w_j}^{\sqrt{\delta}}}\|_{p/2} \lesssim_\epsilon \delta^{-q'}$~~ $\|\sum 1_{T_{w_j}^{\sqrt{\delta}}}\|_{q'} \le C_\epsilon \delta^{-\frac{1}{2}(\frac{d}{q}-1+\epsilon)}$.

observe the $\sqrt{\delta}$!

Rescaling by δ^{-1} (i.e., replacing $T_{w_j}^{\sqrt{\delta}}$ by τ_j with dimensions $R^{\frac{1}{2}}\times R^{\frac{1}{2}}\times R = \delta^{-1/2}\times\delta^{-1/2}\times\delta^{-1}$)

gives $\|\sum 1_{\tau_j}\|_{q'} \le \delta^{-\frac{1}{2}(\frac{d}{q}-1+\epsilon)}\cdot\delta^{-3/q'} \underset{d=3}{=} \delta^{-1-3/p-\epsilon}$

Combining this with $\|G\|_{L^p(B_0(R))} \lesssim \delta^{3/4+1/2p}\|\sum 1_{\tau_j}\|^{1/2}_{p/2}$ shows

$$\|G\|^p_{L^p(B_0(R))} \lesssim \delta^{\frac{3p}{4}+\frac{1}{2}-\frac{p}{2}-\frac{3}{2}-\epsilon\frac{p}{2}} = \delta^{p/4-1-\epsilon}$$

$$\Rightarrow \|(f d\sigma)^\vee\|_{L^p(B_0(R))} \lesssim \delta^{1/4-1/p-\epsilon} = R^{-\frac{1}{4}+\frac{1}{p}+\epsilon}$$

~~$p \ge 4$~~ ok if $p < 4$

$R^{-\frac{1}{4}+\frac{1}{p}}$ also beats the $R^{\frac{2}{p}-\frac{1}{2}}$ from interpol if $p < 4$

(T.T)