1) Proof of restriction conjecture for curves in $\mathbb{R}^2$ (Demeter's book (Ch 1)) ✓

2) Equivalence between restriction theorems for paraboloid and sphere (concealed in Tao's 2020 notes, but uses parabolic rescaling → see Demeter's book, Ch. 4 or Bourgain–Demeter) (maybe overkill)

3) Proof of $d=2$ restriction conjecture using bilinear restriction (→ old notes of Tao, but also Demeter's book Ch. 1 and later Ch 4 bilinear → linear reduction) (maybe overkill)

4) WP decomposition following Bourgain–Demeter (not needed here)

nice review by Foschi–Oliveira e Silva

1 Proof of restriction conjecture for curves in $\mathbb{R}^2$

The argument is due to Hörmander.

Let $\phi: [-1,1] \to \mathbb{R}$ be a C^2 fct with $\inf_{|\xi|\le 1} |\phi''(\xi)| \ge \nu > 0$ and for $f: [-1,1] \to \mathbb{C}$ let $(E^\phi f)(x_1,x_2) := \int f(\xi) e^{2\pi i (x_1 \xi + x_2 \phi(\xi))} d\xi$ denote the corresponding extension operator for the curve $(\xi, \phi(\xi))$. The following thm settles the restriction conjecture for such curves, so in particular for S^1 and the truncated paraboloid $P^1 = \{(\xi, \xi^2) : |\xi| < 1\}$

$q' < \frac{4}{3}$ $\quad \frac{d+1}{q} < \frac{d-1}{r'} = 1 - \frac{1}{r}$

Thm 1.1 We have $\|E^\phi f\|_{L^q(\mathbb{R}^2)} \lesssim_\nu \|f\|_{L^r([-1,1])}$ for each $q > 4$ and $\frac{3}{q} + \frac{1}{r} \le 1$

Pf By interpolation, it suffices to check the endpoint $\frac{3}{q} + \frac{1}{r} = 1$, so let $1 \le p < 2$ with $2p' = q$ and $2p/(3-p) = r$. Then let us write

$$\|E^\phi f\|^2_{L^q(\mathbb{R}^2)} = \|E^\phi f \, E^\phi f\|_{L^{p'}(\mathbb{R}^2)}$$

and change variables $(t,s) = T(\xi_1, \xi_2)$ where $(\xi_1,\xi_2) = (\overbrace{t+s}^{\in[-1,1]}, \phi(t) + \phi(s))$. Then

$$(E^\phi f)(x_1,x_2)^2 = \int_{-1}^{1} d\underset{t}{\xi_1} \int_{-1}^{1} d\underset{s}{\xi_2} \, \exp(2\pi i [x_1(\underset{t}{\xi_1} + \underset{s}{\xi_2}) + x_2(\underset{\phi(t)}{\phi(\xi_1)} + \underset{\phi(s)}{\phi(\xi_2)})]) f(\underset{t}{\xi_1}) f(\underset{s}{\xi_2})$$

$$\underset{\text{symmetrize}}{=} 2 \iint_{t>s} dt\, ds\, \exp(2\pi i [x_1(t+s) + x_2(\phi(t) + \phi(s))]) f(t) f(s)$$

$$\underset{\text{change of variable}}{=} \iint d\xi_1 d\xi_2 \, \exp(2\pi i [x_1 \xi_1 + x_2 \xi_2]) (f \otimes f)(T(\xi_1,\xi_2)) \cdot |\det T'(\xi_1,\xi_2)|$$

$\Rightarrow (E^\phi f)(x_1,x_2)^2$ is the FT of $F(\xi_1,\xi_2) := (f\otimes f)(T(\xi_1,\xi_2)) \cdot |\det T'(\xi_1,\xi_2)|$

$\Rightarrow$ by Hausdorff–Young, $\|(E^\phi f)^2\|_{L^{p'}(\mathbb{R}^2)} \le \|F\|_{L^p(\mathbb{R}^2)}$, so it remains to show that $\|F\|_{L^p(\mathbb{R}^2)} \lesssim_\nu \|f\|^2_{L^r([-1,1])}$.

We shall use $(T^{-1})' = \begin{pmatrix} 1 & 1 \\ \phi'(t) & \phi'(s) \end{pmatrix}$ (in t,s-coords) $\to |\det (T^{-1})'| = |\phi'(t) - \phi'(s)|$

To that end, we use the old variables (t,s), use $|\phi'(t)-\phi'(s)| \gtrsim \nu |t-s|$, and Hölder, to obtain

$$\|F\|^p_{L^p(\mathbb{R}^2)} = \int dt\,ds\ |f(t)f(s)|^p \frac{|\det(T^{-1})|}{|\det(T^{-1})|^p} = \int dt\,ds\ |f(t)f(s)|^p \frac{1}{|\phi'(t)-\phi'(s)|^{p-1}}$$

$$\lesssim_\nu \iint |f(t)f(s)|^p \frac{1}{|t-s|^{p-1}} \le \|f^p\|_{r/p} \underbrace{\| |f|^r * |\cdot|^{1-p} \|_{(r/p)'}}_{\overset{HLS}{\lesssim} \|f^p\|_{r/p}} \lesssim \|f\|_r^{2p}$$

$\boxed{2}$ $\square$

2 Warm-up for the proof of 2d-restriction using bilinear techniques extensively

(→ Demeter Prop 1.19, Lemma 1.20) Recall that for two Borel measures ν_1, ν_2 on $\mathbb{R}^d$, we have

$$(\nu_1 * \nu_2)(A) := \iint_{\mathbb{R}^d \mathbb{R}^d} \mathbb{1}_A(\xi+\eta)\,d\nu_2(\eta)\,d\nu_1(\xi)$$

$$(\nu_1 * \nu_2)(x) = \iint \delta(x-\xi-\eta)\,d\nu_1(\xi)\,d\nu_2(\eta)$$

~~Prop 2.1 Let μ be a pos~~

Prop 2.1 Let $\mu \ge 0$ be a finite Borel measure on $\mathbb{R}^d$ s.t. $\mu * \mu = F(\xi)\,d\xi$ for some $F \in L^\infty(\mathbb{R}^d, d\xi)$

supported on $\mathbb{R}^d + \mathbb{R}^d = \{x+y : x \in \mathbb{R}^d, y \in \mathbb{R}^d\} = \mathbb{R}^d$

i.e., $\mu * \mu$ is ac wrt Lebesgue, $\mu * \mu \ll d\xi$. (usually the case when curvature is present so that distinct portions of a surface become transverse to each other)

$$\Rightarrow \|\widehat{g\,d\mu}\|_{L^4(\mathbb{R}^d)} \lesssim \|g\|_{L^2(d\mu)} = \sqrt{\int |g(x)|^2\,d\mu(x)}$$

Pf Let $\phi \in C^\infty(\mathbb{R}^d : [0,\infty))$ s.t. $\int \phi = 1$ and $\phi_\epsilon(x) := \epsilon^{-d}\phi(x/\epsilon)$, and define the regularization

$$\mu_\epsilon(\xi) := (\phi_\epsilon * \mu)(\xi) = \int_{\mathbb{R}^d} \phi_\epsilon(\xi-\eta)\,d\mu(\eta).$$

Obviously $\mu_\epsilon \in L^1(\mathbb{R}^d)$ since $\int d\xi \int d\mu(\eta)\,\phi_\epsilon(\xi-\eta) = \overset{=1}{\|\phi\|_{L^1(d\xi)}}\,\mu(\mathbb{R}^d) < \infty$. and

for $f \in C(\mathbb{R}^d)$ (or $f \in L^\infty$ even) $\int d\xi\, f(\xi) \int d\mu(\eta)\,\phi_\epsilon(\xi-\eta) = \int d\mu(\eta)\,\underbrace{(f * \phi_\epsilon)(\eta)}_{\xrightarrow{\text{ptwise a.e.}} f(\eta)}$

$$\to \int d\mu(\eta)\, f(\eta),$$

so $\mu_\epsilon\,d\xi \to d\mu$.

$\Rightarrow$ It suffices to prove the claim for μ_ϵ instead of μ with implicit constant independent of ϵ, i.e., for $G \in L^2(\mathbb{R}^d, d\xi)$,

$$\cancel{\|\widehat{G\mu_\epsilon}\|_{L^4(\mathbb{R}^d, d\xi)} \lesssim \|G\|_{L^2(\mathbb{R}^d, d\mu_\epsilon)} = \sqrt{\int |G(\xi)|^2\,d\mu(\xi)}}$$

$$\|\widehat{G\mu_\epsilon}\|_{L^4(\mathbb{R}^d, d\xi)} = \left(\int |\widehat{G\mu_\epsilon}(x)|^4\,dx\right)^{1/4} \lesssim \left(\int |G(x)|^2 \mu_\epsilon(x)\,dx\right)^{1/2}$$

③

$$\|\widehat{G\mu_\epsilon}\|^4_{L^4(\mathbb{R}^d,d\xi)} = \left(|\widehat{G\mu_\epsilon}|^2, |\widehat{G\mu_\epsilon}|^2\right) = \left(\mathcal{F}^{-1}(|\widehat{G\mu_\epsilon}|^2), \mathcal{F}^{-1}(|\widehat{G\mu_\epsilon}|^2)\right)$$

$$= \int |G\mu_\epsilon * G\mu_\epsilon|^2 d\xi = \int_{\mathbb{R}^d} d\xi \left|\int d\eta\, G(\xi-\eta)G(\eta)\mu_\epsilon(\xi-\eta)\mu_\epsilon(\eta)\right|^2$$

$$\overset{CS}{\leq} \int d\xi \left(\int d\eta\, |G(\xi-\eta)|^2|G(\eta)|^2 \mu_\epsilon(\xi-\eta)\mu_\epsilon(\eta)\right)\underbrace{\left(\int d\eta\, \mu_\epsilon(\xi-\eta)\mu_\epsilon(\eta)\right)}_{\leq \|\mu_\epsilon * \mu_\epsilon\|_{L^\infty(\mathbb{R}^d)}}$$

$$\leq \|\mu_\epsilon * \mu_\epsilon\|_{L^\infty(\mathbb{R}^d)} \underbrace{\left(\int |G(\xi)|^2\mu_\epsilon(\xi)d\xi\right)^2}_{\text{what we want}}$$

$$* = \|(\mu * \phi_\epsilon)*(\mu*\phi_\epsilon)\|_{L^\infty} \leq \|\mu*\mu\|_{L^\infty}\|\phi_\epsilon * \phi_\epsilon\|_{L^1} = \|F\|_{L^\infty} < \infty$$

$$\mu * \mu = F d\xi, \quad F \in L^\infty.$$

Surface measure of S^{d-1} satisfies hypothesis of Prop 2.1

Lemma 2.2 Let $d\sigma$ be surface measure of S^{d-1}, then for each $d \geq 2$, the measure $d\sigma * d\sigma$ is ac wrt Lebesgue, i.e., $d\sigma * d\sigma = F d\xi$ for some $F \in L^\infty(\mathbb{R}^{d+1})$.

Moreover, $F(\xi) = 0$ for $|\xi| > 2$ and satisfies

$$|F(\xi)| \leq \begin{cases} |\xi|^{-1} & 0 < |\xi| \leq 1 \\ (2-|\xi|)^{(d-3)/2}, & 1 \leq |\xi| \leq 2 \end{cases}$$

$d\sigma * d\sigma$ supported on $S^{d-1} + S^{d-1} \subseteq \mathbb{R}^d$
$= \{x+y : x, y \in S^{d-1}\}$

Remark Explicit computation (see, e.g., (3.2) in Foschi – Diogo Oliveira E Silva), one finds

$$(\sigma_{d-1} * \sigma_{d-1})(\xi) = 2\int_{S^{d-1}} \delta(1-|\xi-\omega|^2)d\sigma_\omega = \frac{2}{|\xi|}\int_{S^{d-1}} \delta\left(2\frac{\xi}{|\xi|}\cdot\omega - |\xi|\right)d\sigma_\omega = \frac{|S^{d-2}|}{|\xi|}\left(1-\frac{|\xi|^2}{4}\right)_+^{\frac{d-3}{2}}$$

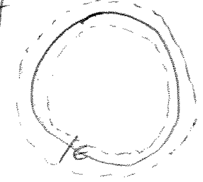
$\Rightarrow$ Prop 2.1 not directly applicable!

[One uses $\sigma_{d-1}(\xi) = \frac{1}{2}\delta(|\xi|^2-1)\, d\xi \to$ $\quad \epsilon^{-1}\chi\left(\frac{|\xi|^2-1}{\epsilon}\right)$

$S = \{\xi \in \mathbb{R}^d : |P(\xi)| = 0\}$ with Leray measure $d\Sigma_S(\xi) = \frac{d\sigma(\xi)}{|\nabla P(\xi)|} = \delta(P(\xi))d\xi$

$$\iint \delta(\xi-\eta_1-\eta_2)\,\delta(|\eta_1|^2-1)\,\delta(|\eta_2|^2-1)\,d\eta_1 d\eta_2 = \int d\eta_2\, \delta(|\eta_2|^2-1)\,\delta(1-|\xi-\eta_2|^2) = \int_{S^{d-1}} d\Sigma_{S^{d-1}}(\eta_2)\,\delta(1-|\xi-\eta_2|^2)$$

$\eta_1 = \xi - \eta_2, \quad |\xi-\eta_2|^2 = 1$

Pf 

Let S^{d-1}_ϵ be the ϵ-neighborhood of S^{d-1} and $\sigma_\epsilon := \epsilon^{-1}\mathbb{1}_{S^{d-1}_\epsilon}$. Then $\sigma_\epsilon d\xi \to d\sigma$ in $\mathbb{R}^d$

since $d\Sigma(\xi) = \frac{d\sigma}{|\nabla P(\xi)|} = \frac{d\sigma(\xi)}{2} = \delta(P(\xi)) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\chi\left(\frac{|\xi|^2-1}{\epsilon}\right)$

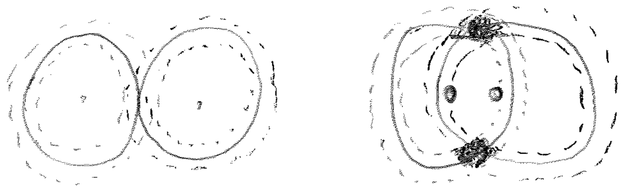
[$\sim$ approx. of identity since $\frac{1}{\epsilon}\int \chi\left(\frac{|\xi|-1}{\epsilon}\right)d\xi = \frac{1}{\epsilon}\int_{1/2}^{3/2} dh\, h^{d-1}\chi\left(\frac{h-1}{\epsilon}\right) = \frac{1}{\epsilon}\int_{-1/2}^{1/2} dh\,(h+1)^{d-1}\chi(h/\epsilon)$

$\sim \frac{1}{\epsilon}\int_{-1/2}^{1/2} dh\, \chi(h/\epsilon) \sim 1$

Next note $(\sigma_\epsilon * \sigma_\epsilon)(\xi) = \frac{1}{\epsilon^2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \delta(\xi-\eta_1-\eta_2)\mathbb{1}_{S^{d-1}_\epsilon}(\eta_1)\mathbb{1}_{S^{d-1}_\epsilon}(\eta_2)\,d\eta_1 d\eta_2$

$$= \epsilon^{-2}\left|S^{d-1}_\epsilon \cap (\xi + S^{d-1}_\epsilon)\right|$$

$\sigma_\epsilon * \sigma_\epsilon(\xi) = \epsilon^{-2} \, | S_\epsilon^{d-1} \cap (\xi + S_\epsilon^{d-1})| \neq 0$ only if $|\xi| < 2 + 2\epsilon$



now let it spin

$\Rightarrow S_\epsilon^{d-1} \cap \xi + S_\epsilon^{d-1}$ is a body of revolution

$\Rightarrow$ volume given by at a most a constant multiple of the area of the cross section $S_\epsilon^{d-1} \cap ((r,0) + S_\epsilon^{d-1})$ where $r = |\xi|$, with $d = 2$

Suppose first $0 < r < 1$. Then note that any $y = (y_1, y_2) \in S_\epsilon^{d-1} \cap (\xi + S_\epsilon^{d-1})$ with $\xi = (r, 0)$ satisfies

(i) $1 - 2\epsilon \le y_1^2 + (y_2)^2 \le 1 + 3\epsilon$ since $y \in S_\epsilon^1$ and $y \in \xi + S_\epsilon^1$ with $\xi = (r, 0)$, $r < 1$

and (ii) $1 - 2\epsilon \le (y_1 - r)^2 + y_2^2 \le 1 + 3\epsilon$ by symmetry

$\Rightarrow$ Combining $y_1^2 + y_2^2 \ge 1 - 2\epsilon$ from (i) and $-2y_1 r + r^2 + \underbrace{y_1^2 + y_2^2}_{\ge 1 - 2\epsilon} \le 1 + 3\epsilon$ from (ii), one obtains

$|2y_1 - r| \le \frac{5\epsilon}{r}$.

$\Rightarrow$ Horizontal projection of $S_\epsilon^1 \cap ((r,0) + S_\epsilon^1)$ sits inside an horizontal ~~step~~ interval of length $5\epsilon/r$.

Since $r \le 1$, the vertical slices of $S_\epsilon^1 \cap ((r,0) + S_\epsilon^1)$ have length $\approx \epsilon$

$\Rightarrow |S_\epsilon^1 \cap ((r,0) + S_\epsilon^1)| \lesssim \epsilon^2 / r$ (by Fubini) and so, going back to d dimensions,

$\sigma_\epsilon * \sigma_\epsilon(\xi) = \epsilon^{-2} | S_\epsilon^{d-1} \cap (\xi + S_\epsilon^{d-1})| \lesssim r^{-1}$ for $|\xi| \le 1$.

Similar arguments show that for $1 < |\xi| \le 2$, $\sigma_\epsilon * \sigma_\epsilon \lesssim (2 - |\xi|)^{(d-3)/2}$, $0 < \epsilon < 2$.

Since $(\sigma_\epsilon * \sigma_\epsilon)\, d\xi \to d\sigma * d\sigma$ weakly, i.e. when integrated against $L^1(\mathbb{R}^d)$ fcts,

$d\sigma * d\sigma \ll d\xi$ with $\frac{d\sigma * d\sigma}{d\xi} = F \lesssim |\xi|^{-1} \mathbb{1}_{|\xi| < 1} + (2 - |\xi|)^{(d-3)/2} \mathbb{1}_{1 < |\xi| < 2}$.

3 Proof of two-dimensional restriction conjecture using bilinear restriction (originally by Córdoba–Fefferman)

We already observed in Thm 1.1 and Prop 2.1 that, whenever even Lebesgue exponents are involved, we can write $\|F\|_{L^4}^2 = \|F \cdot F\|_{L^2} = \|\hat{F} * \hat{F}\|_{L^2}$ and suddenly the role of oscillations / cancellation and pure size estimates + precise knowledge of geometry, in terms of supports of involved functions / measures, are interchanged.

Recall we want to prove $\|(g d\sigma)^\wedge\|_{L^q(\mathbb{R}^2)} \lesssim \|f\|_p$ for $q > \frac{2d}{d-1}$ and $q \geq \frac{(d+1)p'}{d-1}$.

By interpolation, it suffices to treat $q = 4+\epsilon$; in fact we shall use Marcinkiewicz and ($q=\infty$ obvious)

prove $\mathcal{F}_S^* : L^{p'} \to L^{q,\infty}$, i.e., $\|(\mathbb{1}_\Omega d\sigma)^\wedge\|_{L^{q,\infty}} = \lambda |\{ |(\mathbb{1}_\Omega d\sigma)^\wedge(x)| \geq \lambda \}|^{1/q} \lesssim \|\mathbb{1}_\Omega\|_{L^p} = |\Omega|^{1/p}$ $\forall \Omega \subseteq S^1$

(cf. old HA notes)

Actually we shall prove the slightly stronger $L^{p'} \to L^q$ estimate ($q > 4$)
$\|(\mathbb{1}_\Omega d\sigma)^\wedge\|_q \lesssim |\Omega|^{1/p}$, $\Omega \subseteq S^1$ (actually, to ease the analysis, we consider only quadrant of S^1 to avoid nuisances with antipodal points)

Now let us as before square the estimate and make use of transversality between ~~distant~~ different chunks of S^1

$$\|(\mathbb{1}_\Omega d\sigma)^\wedge\|_q^2 = \|(\mathbb{1}_\Omega d\sigma)^\wedge (\mathbb{1}_\Omega d\sigma)^\wedge\|_{q/2} \overset{?}{\lesssim} |\Omega|^{2/p} \;? \quad q>4 \to \tfrac{q}{2} > 2$$

So we shall prove bounds on $\|(f d\sigma)^\wedge (g d\sigma)^\wedge\|_2$ and $\|(f d\sigma)^\wedge (g d\sigma)^\wedge\|_\infty$ and interpolate between them. (f, g being arbitrary complex fcts. on S^1).

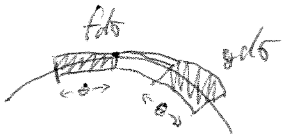
The latter quantity is more accessible and we obtain $\|(f d\sigma)^\wedge\|_\infty \leq \|f\|_{L^1(S^1)}$.
$\rightarrow \|(f d\sigma)^\wedge (g d\sigma)^\wedge\|_\infty \leq \|f\|_{L^1} \|g\|_{L^1}$

To estimate $\|(f d\sigma)^\wedge (g d\sigma)^\wedge\|$ we shall decompose $(f d\sigma)$ and $g d\sigma$ into smaller chunks on S^1 ~~and see th~~ $\Rightarrow$ There will be a lot of "off diagonal" terms which only "interact weakly" with each other and few "on diagonal" terms which interact strongly with each other.

~~Before~~ $\rightarrow$ 3 steps:
- decompose (dyadically)
- estimate interaction
- glue everything together and sum up.

Note that it would have been nice to use Lem 2.2 and apply Prop 2.1 to prove Restriction conj. However the $|\xi|^{-1}$ singularity of $(d\sigma * d\sigma)(\xi)$ is necessary; it's essentially due to high symmetry of S^{d-1}, i.e., the fact that 0 can be represented in multiple ways by $\xi + \eta$, $\xi, \eta \in S^{d-1}$; for this reason we decompose the S^1

We start with the second item and assume first that $f\,d\sigma$ and $g\,d\sigma$ are supported on arcs of length θ which are also seperated from each other by θ



$|\operatorname{supp} f d\sigma| \sim \theta, \quad |\xi - \eta| \sim \theta \quad \forall\, \xi \in \operatorname{supp} f d\sigma, \ \eta \in \operatorname{supp} g d\sigma$

<u>Lemma 3.1</u> Suppose f and g are supported on distinct θ-arcs of S^1, say I_1, I_2, whose separation is also comparable to θ.

$$\Rightarrow \quad \| (f d\sigma)^\wedge (g d\sigma)^\wedge \|_{L^2} \lesssim \theta^{-1/2} \|f\|_2 \|g\|_2 .$$

(cf. Prop 2.1, here finer) Lem 2.2 $\to$ $|\xi|^{-1}$ sing in deadl gets resolved ~~ther~~ here.

<u>Pf</u> By Plancherel, it suffices to show $\underbrace{\| f d\sigma * g d\sigma \|_2}_{\text{supported on } S^1 + S^1} \lesssim \theta^{-1/2} \|f\|_2 \|g\|_2$

To prove it, we interpolate between $L^1 \to L^1$ and $L^\infty \to L^\infty$

$$\underbrace{\| f d\sigma * g d\sigma \|_{L^1}} \lesssim \|f\|_1 \|g\|_1 \qquad \| f d\sigma * g d\sigma \|_\infty \lesssim \theta^{-1} \|f\|_\infty \|g\|_\infty .$$

$$\le \int_{S^1} d\sigma(\eta_1) \int_{S^1} d\sigma_2(\eta_2)\, f(\eta_1) g(\eta_2) \underbrace{\mathbb{1}_{S^1+S^1}(\eta_1+\eta_2)}_{\le 1}$$

$$\le \|f\|_1 \|g\|_1$$

So we are left to show that $(f d\sigma * g d\sigma)(\xi) = \int_{I_1} d\sigma(\eta_1) \int_{I_2} d\sigma(\eta_2)\, \delta(\xi - \eta_1 - \eta_2) f(\eta_1) g(\eta_2)$

$$\lesssim \theta^{-1} \|f\|_\infty \|g\|_\infty .$$

As before, we regularize $d\sigma_\xi|_{I_1} \equiv d\sigma_I$ by convolving it with an approximation of the identity $\phi_\epsilon = \epsilon^{-1} \mathbb{1}_{I_1^\epsilon}$ where I_1^ϵ is ~~an~~ ϵ-neighborhood of I_1, i.e.,

$$I^\epsilon = \{ r(\cos\theta, \sin\theta) : (\cos\theta, \sin\theta) \in I,\ |r-1| < \tfrac{1}{2}\epsilon \}$$

Then as before the Lebesgue measure $d\xi|_{I^\epsilon} \xrightarrow{\epsilon \to 0} d\sigma_I$ and

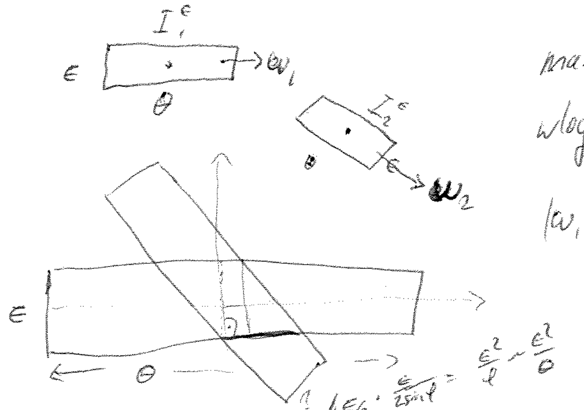
$$\to \ f d\sigma_{I_1} * g d\sigma_{I_2} = \text{w-}\lim_{\epsilon \to 0} \underbrace{f\, d\sigma_{I_1^\epsilon} * g\, d\sigma_{I_2^\epsilon}}$$

$$\frac{1}{4\epsilon^2} \int_{\mathbb{R}^2} f(\xi-\eta) \mathbb{1}_{I_1^\epsilon}(\xi-\eta) g(\eta) \mathbb{1}_{I_2^\epsilon}(\eta)\, d\eta \ \le \|f\|_\infty \|g\|_\infty \frac{1}{4\epsilon^2} \int_{\mathbb{R}^2} d\eta\, \mathbb{1}_{I_1^\epsilon} * \mathbb{1}_{I_2^\epsilon}$$

$\to$ remains to show $\frac{1}{\epsilon^2} \underbrace{\| \mathbb{1}_{I_1^\epsilon} * \mathbb{1}_{I_2^\epsilon} \|_\infty}_{\sup_\xi |I_1^\epsilon \cap (\xi + I_2^\epsilon)|} \lesssim \theta^{-1}$, uniformly in $\epsilon > 0$.

<u>HW?</u> ~~Also bec~~ This is due to the geometric fact that any translate of I_1^ϵ intersects I_2^ϵ in an arc of measure at most $\epsilon^2 \theta^{-1}$.

We ~~make~~ derive this bound explicitly for the case where I_1^ϵ and I_2^ϵ are replaced by unbent rectangles (This is heuristically justified for $\theta \ll 1$ which is obviously the interesting case.)



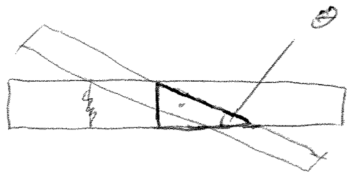
maximal overlap if they sit on top of each other.

wlog $\omega_1 = (1,0)$, $\omega_2 = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \sim \begin{pmatrix} 1 \\ \varphi \end{pmatrix}$ $\quad 0<\varphi\ll 1$, $\varphi \sim \theta$

$|\omega_1 - \omega_2| = \left\| \begin{pmatrix} 1-\cos\varphi \\ -\sin\varphi \end{pmatrix} \right\| \sim \varphi$

area of parallelogram
= base × height
= base × ϵ

(base $=\theta$ if $\varphi=0$
base $=\epsilon$ if $\varphi=\frac{\pi}{2}$
base $=$? if $\varphi=\theta$



$\tan\theta \sim \sin\theta \sim \theta$
ϵ / multiple of base
$\to$ multiple of base $\sim \frac{\epsilon}{\theta}$

Remember we want to show $\| \widehat{\mathbb{1}_\Omega d\sigma}\, \widehat{\mathbb{1}_\Omega d\sigma} \|_{q/2} \lesssim |\Omega|^{2/p}$, $\quad q>4$, $p' \le q/3$

we know: $\| \widehat{f d\sigma}\, \widehat{g d\sigma} \|_\infty \le \|f\|_1 \|g\|_1 \quad \forall f,g$

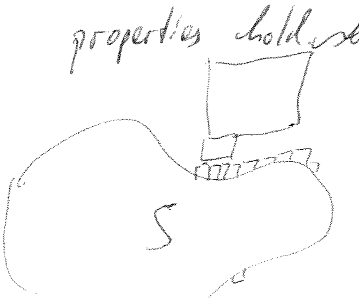
$\| \widehat{f d\sigma_{I_1}}\, \widehat{g d\sigma_{I_2}} \|_2 \lesssim \|f\|_2 \|g\|_2 \cdot \frac{1}{\theta^{1/2}}$ whenever I_1, I_2 are two θ-arcs that are θ-separated from each other.

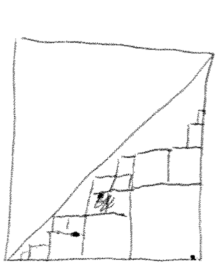
Recall the dyadic Whitney decomposition

Prop Let $S \subseteq \mathbb{R}^d$ be a closed set $\Rightarrow \exists$ collection $\mathcal{C}$ of closed dyadic cubes Ω with pairwise disjoint ~~support~~ interiors s.t. $\mathbb{R}^d \setminus S = \bigcup_{\Omega\in\mathcal{C}} \Omega$ and

$$\underbrace{4\,\ell(\Omega)}_{\text{side length}} \le \operatorname{dist}(\Omega, S) \le 50\,\ell(\Omega)$$

Pf Let $\mathcal{C}'$ be the collection of all dyadic cubes in $\mathbb{R}^d$ s.t. $8\Omega \cap S = \emptyset$. Let $\mathcal{C}$ consist of those cubes in $\mathcal{C}'$ that are maximal with respect to inclusion $\to$ the desired properties hold ~~so~~ since dyadic cubes either contain each other or are disjoint ☐

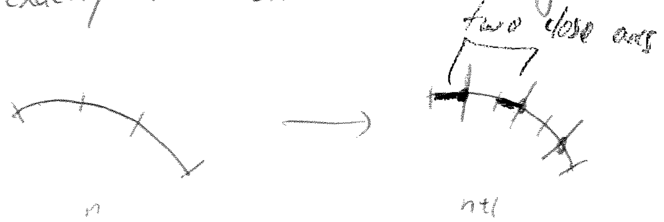




decomposing $[-1,1]^2 \setminus$ diagonal

Now let's adapt the Whitney decomposition to our quarter circle.

For every $n \geq 0$ divide S^1 into 2^n equal arcs so that each arc at stage n has exactly two children at stage $n+1$. The set of arcs at stage n is denoted by A_n.



We say that two arcs in A_n are close if they are not adjacent, but their parents are. If I and J are close, we write $I \sim J$.

Note that for every $x, y \in S^1$, there is exactly one pair of arcs I and J containing x and y respectively, and satisfying $I \sim J$

$$\Rightarrow (\mathbb{1}_\Omega d\sigma)^\wedge (\mathbb{1}_\Omega d\sigma)^\wedge = \sum_{I \sim J} (\mathbb{1}_\Omega d\sigma_I)^\wedge (\mathbb{1}_\Omega d\sigma_J)^\wedge = \sum_{n \geq 1} \sum_{\substack{I, J \in A_n \\ I \sim J}} (\mathbb{1}_\Omega d\sigma_I)^\wedge (\mathbb{1}_\Omega d\sigma_J)^\wedge$$

and note that for each fixed J, there are only $O(1)$ many I close to J. (indep of n)

$\rightarrow$ to deal with the n-summation, we use triangle ineq. (oscillations hidden in convolution where geometry was exploited quite well)

$$\Rightarrow \|(\mathbb{1}_\Omega d\sigma)^\wedge (\mathbb{1}_\Omega d\sigma)^\wedge\|_{q/2} \lesssim \sum_{n \geq 1} \Big\| \sum_{\substack{I, J \in A_n \\ I \sim J}} (\mathbb{1}_\Omega d\sigma_I)^\wedge (\mathbb{1}_\Omega d\sigma_J)^\wedge \Big\|_{q/2}$$

Now we use our previous $1 \to \infty$ and $2 \to 2$ bounds and interpolate also for $\sum_{I,J}$ -sum and obtain

For $1 \to \infty$-bound, we use triangle ineq.

$$\Big\| \sum_{\substack{I, J \in A_n \\ I \sim J}} (\mathbb{1}_\Omega d\sigma_I)^\wedge (\mathbb{1}_\Omega d\sigma_J)^\wedge \Big\|_\infty \leq \sum_{\substack{I, J \in A_n \\ I \sim J}} \|\mathbb{1}_\Omega d\sigma_I\|_{L^1} \|\mathbb{1}_\Omega d\sigma_J\|_{L^1} \leq \sum_{\substack{I, J \in A_n \\ I \sim J}} |\Omega \cap I| \cdot |\Omega \cap J|$$

Although it would be nice to have more tractable estimates involving $|\Omega|$ or 2^{-n} on RHS, crude estimates suffice for our purposes. We use $|\Omega \cap I| \leq |I| = 2^{-n}$ and exploit that only $O(1)$ many I and J are close and obtain

$$\Big\| \sum_{\substack{I, J \in A_n \\ I \sim J}} (\mathbb{1}_\Omega d\sigma_I)^\wedge (\mathbb{1}_\Omega d\sigma_J)^\wedge \Big\|_\infty \lesssim \sum_{\substack{I, J \in A_n \\ I \sim J}} |\Omega \cap I| \cdot 2^{-n} \lesssim \sum_{I \in A_n} |\Omega \cap I| \, 2^{-n} = 2^{-n} |\Omega|$$

Alternatively, we may simply drop the $I \sim J$ condition in the summation, and obtain $\| \sum_{\substack{I, J \in A_n \\ I \sim J}} (1_\Omega d\sigma_I)^\wedge (1_\Omega d\sigma_J)^\wedge \|_\infty \le \sum_{I \in A_n} |I \cap \Omega| \sum_{J \in A_n} |J \cap \Omega| = |\Omega|^2$

and so ~~it~~ combining them gives

$$\to (*) \quad \| \sum_{\substack{I, J \in A_n \\ I \sim J}} (1_\Omega d\sigma_I)^\wedge (1_\Omega d\sigma_J)^\wedge \|_\infty \lesssim |\Omega| \min(|\Omega|, 2^{-n})$$

Now for the L^2 estimate simply using triangle ineq would be a bad idea as they're perfect for ~~as~~ exploiting orthogonality.

Key observation (Fefferman): As $I \sim J$ vary, the functions $(1_\Omega d\sigma_I)^\wedge (1_\Omega d\sigma_J)^\wedge$ have essentially disjoint Fourier supports ($1_\Omega d\sigma_I * 1_\Omega d\sigma_J$ for different I and J are ~ disjointly supported) ~ $1_\Omega d\sigma_I * 1_\Omega d\sigma_J$ are almost orthogonal for different pairs (I, J) as the ~~set theo~~ Minkowski sums $I + J$ are almost disjoint for different pairs (I, J).

In fact, one can achieve perfect orthogonality by only considering every tenth pair, say $(I \sim J)$ and then adding up the ten smaller sums by the triangle inequality.

$$\Rightarrow \| \sum_{I, J \in A_n, I \sim J} (1_\Omega d\sigma_I)^\wedge (1_\Omega d\sigma_J)^\wedge \|_2^2 \lesssim \sum_{I, J \in A_n, I \sim J} \| (1_\Omega d\sigma_I)^\wedge (1_\Omega d\sigma_J)^\wedge \|_2^2$$

$$\overset{\text{Lem 3.1}}{\lesssim} \sum_{I, J \in A_n, I \sim J} 2^n |I \cap \Omega| |J \cap \Omega|$$

$\theta \sim 2^{-n}$

$$\overset{\text{previous reasoning}}{\lesssim} 2^n |\Omega| \min(|\Omega|, 2^{-n})$$

$\Rightarrow$ Interpolating this with $(*)$ gives $\| \sum_{I, J \in A_n, I \sim J} (1_\Omega d\sigma_I)^\wedge (1_\Omega d\sigma_J)^\wedge \|_{q/2} \lesssim 2^{2n/q} (|\Omega| \min(|\Omega|, 2^{-n}))^{1 - \frac{2}{q}}$

(don't forget taking $\sqrt{\ }$ of $L^2 \to L^2$ - bound)

and summing this over $n \ge 1$ gives $\| (1_\Omega d\sigma_{S^1})^\wedge (1_\Omega d\sigma_{S^1})^\wedge \|_{q/2} \lesssim \sum_{n \ge 1} 2^{2n/q} (|\Omega| \min(|\Omega|, 2^{-n}))^{1 - \frac{2}{q}}$

$\sim \dots \sim |\Omega|^{1 - 2/q} = |\Omega|^{2/p}$,

as desired

$\square$