The restriction problem

Stating the conjecture + wave packets

Recall that $\mathcal{F}f \in L^2$ whenever $f \in L^2$, hence $\hat{f}$ belongs to an equivalence class within which its members are allowed to differ off of sets of measure zero. In particular a restriction of $\hat{f}$ to codimension $k \geq 1$ submanifolds embedded in $\mathbb{R}^d$ is meaningless. On the other hand we know that when f decays sufficiently fast, then $\hat{f}$ maybe smooth enough to be meaningfully restricted

Example: $f \in L^1 \to \hat{f} \in C_\infty$, i.e., bdd + continuous + decaying at infinity.

Here we shall focus on codimension-one ~~ne~~ submanifolds S and ask for conditions (sufficient and necessary) under which $\hat{f}|_S$ is meaningfully defined.

Recall the classic trace lemma. To that end suppose that we are given a function $F: \underset{\mathbb{R}^{d-1}}{\underset{\cap}{\Omega}} \to \mathbb{R}$ continuous where $\Omega \subseteq \mathbb{R}^{d-1}$ open which parametrizes a surface

(e.g. $x_3 = \pm\sqrt{1 - x_1^2 - x_2^2}$, upper/lower hemisphere)

Prop 4.1 Let $\alpha > 1/2$, then

$$\underbrace{\int_\Omega |\hat{u}(\xi', F(\xi'))|^2 d\xi'}_{\sim \|\hat{u}\|_{L^2(S)}} \lesssim \frac{1}{2\alpha-1} \int_{\mathbb{R}^d} (1+x_d^2)^\alpha |u(x)|^2 dx$$

$$\leq \frac{1}{2\alpha-1} \|u\|^2_{L^2_\alpha}$$

where L^2_α

~~$L^2_\alpha = \{f \in L^2 : \langle x \rangle^{\frac{\alpha}{2}} f \in L^2$~~

$$L^2_\alpha = \{\underbrace{\langle x \rangle^{\alpha/2}}_{=(1+x^2)^{\alpha/4}} f \in L^2\}$$

Pf Let $\tilde{u}(\xi', x_d) = \int_{\mathbb{R}^{d-1}} e^{-2\pi i x' \cdot \xi'} u(x', x_d) dx'$ and

$$\hat{u}(\xi', F(\xi')) = \int_{\mathbb{R}^d} e^{-2\pi i (x' \cdot \xi' + x_d F(\xi'))} u(x', x_d) dx = \int_{\mathbb{R}} dx_d \, e^{-2\pi i x_d F(\xi')} \tilde{u}(\xi', x_d) \frac{\langle x_d \rangle^\alpha}{\langle x_d \rangle^\alpha}$$

By Cauchy-Schwarz, $$|\hat{u}(\xi', F(\xi'))|^2 \leq \underbrace{\left(\int dx_d \langle x \rangle^{-2\alpha}\right)}_{\lesssim \frac{1}{2\alpha-1}} \left(\int dx_d \langle x_d \rangle^{2\alpha} |\tilde{u}(\xi', x_d)|^2\right)$$

$\Rightarrow$ Integrating over $\xi' \in \Omega$ yields $$\int_\Omega |\hat{u}(\xi', F(\xi'))|^2 d\xi' \lesssim \frac{1}{2\alpha-1} \int dx_d \langle x_d \rangle^{2\alpha} \underbrace{\int_\Omega d\xi' |\tilde{u}(\xi', x_d)|^2}$$

$$\leq \int \|\tilde{u}(\cdot, x_d)\|^2_{L^2(\mathbb{R}^{d-1})}$$

$$\overset{\text{Plancherel}}{=} \|u(\cdot, x_d)\|^2_{L^2(\mathbb{R}^{d-1})}$$

$$\Rightarrow \int_\Omega |\hat{u}(\xi', F(\xi'))|^2 d\xi' \lesssim \frac{1}{2\alpha-1} \int dx \langle x_d \rangle^{2\alpha} |u(x', x_d)|^2$$

Prop 4.2 (Hölder continuity of trace lemma)

Let $\alpha > \frac{1}{2}$ and $\theta = \begin{cases} \alpha - 1/2 & \text{if } \alpha < 3/2 \\ 1-\epsilon & \text{for any } \epsilon \in (0,1) \text{ if } \alpha = 3/2 \\ 1 & \text{if } \alpha > 3/2 \end{cases}$

Then $$\int_\Omega |\hat u(\xi', F(\xi')) - \hat u(\xi', \tilde F(\xi')|^2 \, d\xi' \lesssim_{\alpha,\theta} \sup_{\xi' \in \Omega} |F(\xi') - \tilde F(\xi')|^{2\theta} \int_{\mathbb{R}^d} (1+x_d^2)^\alpha |u(x)|^2 dx$$

$\uparrow$ might describe unit sphere

$\uparrow$ might describe sphere with radius $1+\delta$

Pf (HW) With previous notation $\tilde u(\xi', x_d) = \int e^{-2\pi i x' \cdot \xi'} u(x', x_d) \, dx'$ we write

$$|\hat u(\xi', \xi_d) - \hat u(\xi', \tilde\xi_d)|^2 = \left| \int_{\mathbb{R}} dx_d \; \tilde u(\xi', x_d) \left(e^{-2\pi i x_d \xi_d} - e^{-2\pi i x_d \tilde \xi_d} \right) \right|^2$$

$$\overset{CS}{\leq} \underbrace{\left(\int_{\mathbb{R}} dx_d \; \sin^2(\pi(\xi_d - \tilde\xi_d) x_d) \langle x_d \rangle^{-2\alpha} \right)}_{\lesssim |\xi_d - \tilde \xi_d|^{2\theta}} \cancel{\left(\int dx_d \langle x_d \rangle^{2\alpha} \tilde u(\xi', x_d) \right.} \cdot \left(\int_{\mathbb{R}} dx_d \langle x_d \rangle^{2\alpha} |\tilde u(\xi', x_d)|^2 \right)$$

$\to$ Setting $\xi_d = F(\xi')$, $\tilde \xi_d = \tilde F(\xi')$ shows

$$|\hat u(\xi', F(\xi')) - \hat u(\xi', \tilde F(\xi'))| \lesssim |F(\xi') - \tilde F(\xi')|^{2\theta} \int dx_d \langle x_d \rangle^{2\alpha} |\tilde u(\xi', x_d)|^2$$

Integrating over $\xi' \in \Omega$ and applying Plancherel as before shows the claim $\square$

Question Decay can also be measured in term of L^p scales

$\Rightarrow$ can $\hat f$ be meaningfully restricted whenever $f \in L^p$, $1 < p < 2$?

It depends!

Prop 4.3 Let $S = \{\xi \in \mathbb{R}^d : \xi_d = 0, |\xi| \leq 1\}$ equipped with surface measure $d\xi|_S = d\xi'$ and $1 \leq p, q \leq \infty$ and suppose $\|\hat f\|_{L^q(S, d\xi')} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$ holds for all test fcts $f \in \mathcal{S}(\mathbb{R}^d)$. Then $p = 1$

Remark $\|\hat f\|_{L^q(S, d\xi')} \lesssim \|f\|_{L^1}$ $\forall q \in [1, \infty]$ by Hölder and Riemann-Lebesgue.

Pf (HW) Idea: construct a fct f with $\hat f$ concentrating extremely near S. To that end let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and set $f(x) = \varphi(x', x_d/\lambda)$ for $\lambda \gg 1$. $\to \|f\|_{L^p} \sim \lambda^{1/p}$

$\to \hat f(\xi) = \lambda \hat \varphi(\xi', \lambda \xi_d)$ and in particular $\hat f \sim \lambda$ on the pancake $[-1,1]^{d-1} \times [-\frac{1}{\lambda}, \frac{1}{\lambda}]$. In particular $\|\hat f\|_{L^q(S)} \sim \lambda \Rightarrow p \overset{!}{=} 1$ as $\lambda \to \infty$ $\square$

$\Rightarrow$ To meaningfully restrict $\mathbb{R}^d$ FT of L^p-fcts, we <u>need curvature</u>.

Γ $\int_{|\xi'|<1} d\sigma_S(\xi')\, e^{2\pi i x'\cdot\xi'} \equiv F(x', x_d)$ is independent of x_d, i.e. in particular not decaying at all in x_d-direction; F maybe bounded but it's certainly in no L^q $q<\infty$ due to lack of decay.

$\to$ in what follows we shall smooth, compact (later also quadratic, non-compact case) codimension one submanifolds embedded in $\mathbb{R}^d$

with associated Lebesgue surface measure $d\sigma$

$\to$ what are necessary conditions to have $\|\hat{f}\|_{L^q(S,d\sigma)} \lesssim_{p,q,d} \|f\|_{L^p(\mathbb{R}^d)}$?

Before we discuss this in detail, we give an equivalent formulation of restriction estimates, namely (the more common) extension estimates.

<u>Def</u> Let $F_S : \mathcal{S}(\mathbb{R}^d) \to L^\infty(S)$ denote the restriction operator and

$$f \mapsto \hat{f}|_S$$

$$(g, F_S f) = \int g(\xi)(F_S f)(\xi)\, d\sigma(\xi) = \int_S d\sigma(\xi)\, \overline{g(\xi)} \int_{\mathbb{R}^d} dx\, e^{-2\pi i x\cdot\xi} f(x)$$

$F_S^* : C^\infty(S) \to L^\infty(\mathbb{R}^d)$ the extension operator

$$g \mapsto \int d\sigma_S(\xi)\, e^{2\pi i x\cdot\xi} g(\xi) \equiv \widehat{g\, d\sigma}$$

$$\| \quad \int_{\mathbb{R}^d} dx\, f(x) \int_S d\sigma(\xi)\, e^{2\pi i x \xi}\, \overline{g(\xi)}$$

<u>Prop 4.4</u> Thm: Let μ be a ~~finite positive~~ Radon measure on $\mathbb{R}^d$ and $1 \le p, q \le \infty$, ~~with~~

The $\|\hat{f}\|_{L^q(\mathbb{R}^d, d\mu)} \le A \|f\|_{L^p(\mathbb{R}^d)}$, $f \in C_c^\infty(\mathbb{R}^d)$

$$\Leftrightarrow \|\underbrace{(g\, d\mu)^\wedge}_{\int d\mu(\xi)\, e^{2\pi i x\cdot\xi} g(\xi)}\|_{L^{p'}(\mathbb{R}^d)} \le A \|g\|_{L^{q'}(\mathbb{R}^d, d\mu)}, \quad g \in C_c^\infty(\mathbb{R}^d)$$

Moreover, if $q=2$, then the above two statements are also equivalent to $\|\hat{\mu} * \Psi\|_{p'} \le A^2 \|\Psi\|_p$, $\Psi \in C_c^\infty(\mathbb{R}^d)$

<u>Pf</u> Follows from duality. Let $T : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d, d\mu)$, $f \mapsto \hat{f}\, d\mu$, the $T^* : L^{q'}(\mathbb{R}^d, d\mu) \to L^{p'}(\mathbb{R}^d)$

$$|(\underbrace{f}_{\in L^{q'}(\mathbb{R}^d, d\mu)}, \underbrace{T\underbrace{g}_{\in L^p}}_{\in L^q(\mathbb{R}^d, d\mu)})| \le \|f\|_{L^{q'}(\mathbb{R}^d, d\mu)} \|Tg\|_{L^q(\mathbb{R}^d, d\mu)} \le \|f\|_{L^{q'}(d\mu)} \|g\|_{L^p(\mathbb{R}^d)} \|T\|$$

$$= |(T^* f, g)|$$

HW. (cf Prop 8.5.3 in Stein-Shakarchi IV)

④

The ~~first~~ second equivalence $\|\hat\mu * f\|_{L^{p'}} \le A^2 \|f\|_{L^p}$ may need some further explanation since convolutions between measures and $\mathcal{S}$-fcts were not discussed yet

Suppose $\mu \in M(\mathbb{R}^d)$ and $\phi \in C_c^\infty$, then $(\phi * \mu)(x) = \int \phi(x-y)\, d\mu(y) \in C^\infty$ since differentiation under integral sign is justified by dominated convergence.

Usual formulae for FT of fcts extend to measures, i.e.,
$$\widehat{\phi\check\mu} = \hat\phi * \mu \quad \text{if } \phi \in \mathcal{S}$$
$$\widehat{\phi\mu} = \hat\phi * \hat\mu \quad \text{if } \phi \in \mathcal{S}$$

Moreover, there is a kind of Plancherel
$$\int \overline{\hat f}\, \hat g \, d\mu(\xi) = \int (\bar g * \check\mu)(x)\, f(x)\, dx$$

__Thm 4.5__ Let S be a compact, smooth codimension one submanifold with nowhere vanishing gaussian curvature. Suppose $\|\hat f\|_{L^q(S, d\sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$ $\forall f \in \mathcal{S}(\mathbb{R}^d)$.

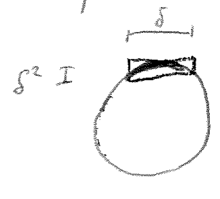
Then (1) $p < \frac{2d}{d+1}$

(2) $\frac{d+1}{p'} \le \frac{d-1}{q}$

__Pf__ To prove (1) we invoke dual formulation $\|(g\, d\sigma)^\wedge\|_{L^{p'}} \lesssim \|g\|_{L^{q'}(S)}$ and take $g \equiv 1$. Then $g \in L^{q'}(S)$ $\forall\, 1 \le q' \le \infty$ but $p' \overset{!}{>} \frac{2d}{d-1}$ since $(d\sigma)^\vee(x) \sim \langle x \rangle^{-(d-1)/2}$.

Statement (2) is due to an examply by A. Knapp. We shall ~~for~~ restrict ourselves to $\mathbb{S}^{d-1}$ for simplicity.

Let $\kappa_\delta = \{ \xi \in \mathbb{S}^{d-1} : 1 - \xi \cdot e_d \le \delta^2 \}$ be a spherical cap located at the north pole with "depth" δ^2 and radius $\sim \delta$ since $|\xi - e_d|^2 = 2(1 - \xi \cdot e_d) \le 2\delta^2$ for $\xi \in \kappa_\delta$



$\rightsquigarrow$ $\Rightarrow |\kappa_\delta| \sim \delta^{d-1}$ ($\mathbb{S}^{d-1}$ smooth, compact)
$\underset{\sigma_{\mathbb{S}^{d-1}}(\kappa_\delta)}{\parallel}$

$\Rightarrow |\xi - e_d| < \sqrt{2}\,\delta$ ~~and~~ for $\xi \in \kappa_\delta$ and $|\xi - e_d| < c^{-1}\delta$ implies $\xi \in \kappa_\delta$

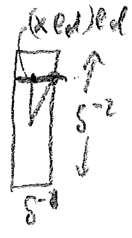
$\Rightarrow \kappa_\delta$ can be thought of intersection $\mathbb{S}^{d-1} \cap R_\delta$ where $R_\delta = \{ \xi \in \mathbb{R}^d : |\xi'| < c\delta,\ |1 - \xi_d| < \delta^2 \}$

Now take $g = \chi_{\kappa_\delta}(\xi) \equiv \chi\left(\frac{\xi - e_d}{\delta}\right)$ where χ is a fixed $C_c^\infty(\mathbb{R}^d)$ bump on $B_0(1)$, i.e., $g \in C_c^\infty(\mathbb{R}^d)$ is a bump adapted to κ_δ. Then g is called a Knapp example at frequency $e_d \in \mathbb{S}^{d-1}$ and spatially centered at origin since no frequency modulation is present. Since $\mathbb{S}^{d-1}$ is smooth and compact, we have $\|g\|_{L^{q'}(\mathbb{S}^{d-1})} \sim \delta^{(d-1)/q'}$. Now we shall compute $\mathcal{F}_S^* g$

$$(g d\sigma)^\wedge(x) = \int_{\kappa_\delta} d\sigma_S(\xi)\ e^{2\pi i x\cdot\xi}\ \chi\left(\frac{\xi-e_d}{\delta}\right)$$

Recall that we assumed $\|(g d\sigma)^\wedge\|_{L^{p'}} \lesssim \|g\|_{L^{q'}(S)}$, so for a necessary condition on q, p, we may estimate the left side from below by merely computing $\|\cdot\|_{L^{p'}(T_{c/\delta})}$ where $T_{c/\delta} = \{x \in \mathbb{R}^d : |x\cdot e_d| \le \left(\frac{c}{\delta}\right)^2$
$|x-(x\cdot e_d)e_d| \le \frac{c}{\delta}\}$

which is the tube dual to the rectangle R_δ that we used to construct κ_δ.



for ~~x~~ $\in T_{c/\delta}$, we estimate $|(g d\sigma)^\wedge(x)| = \left|\int_{\kappa_\delta} d\sigma_S(\xi)\, e^{2\pi i x\cdot\xi}\chi\left(\frac{\xi-e_d}{\delta}\right)\right|$

~~$\gtrsim \int_{\kappa_\delta} d\sigma_S(\xi)\ \cos(2\pi x\cdot\xi)$~~

$$= \left|\int d\sigma_S(\xi)\ \chi\left(\frac{\xi-e_d}{\delta}\right) e^{2\pi i (x-e_d)\cdot\xi}\right|$$

$$\gtrsim \int_{\kappa_\delta} d\sigma_S(\xi)\ \cos(2\pi\, \xi(x-e_d))$$

→ if c in c/δ is so small that $|(x-e_d)\cdot\xi| \le \frac{1}{6}$, say, for all $\xi \in \kappa_\delta$, then $\cos(2\pi\,\xi(x-e_d)) \gtrsim 1$ and accordingly $|(g d\sigma)^\wedge(x)| \gtrsim \frac{1}{2}\sigma_S(\kappa_\delta) \sim \delta^{d-1}$, $x \in T_{c/\delta}$

$\Rightarrow \|(g d\sigma)^\wedge\|_{L^{p'}(T_{c/\delta})} \sim \delta^{d-1}\cdot\delta^{-(d+1)/p'}$

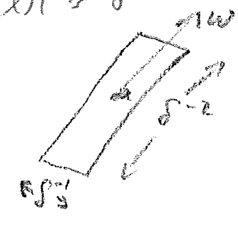
$\Rightarrow$ For $\|(g d\sigma)^\wedge\|_{L^{p'}(T_{c/\delta})} \lesssim \|g\|_{L^{q'}(S)}$ to hold we need $\delta^{d-1-\frac{d+1}{p'}} < \delta^{\frac{d-1}{q'}}$

and hence, since $\delta \ll 1$, $\quad d-1-\frac{d+1}{p'} \ge \frac{d-1}{q'}$, i.e. $\frac{d+1}{p'} \le \frac{d-1}{q}$ $\quad\boxtimes$

Remarks Obviously, κ_δ could have been moved to any other direction $\omega_0 \in S^{d-1}$ without tempering the example. Similarly, g could have been multiplied by a phase $\exp(2\pi i\xi\cdot a)$ which would have only moved the dual tube where $F_S^* g$ is roughly constant.
That is taking ~~g~~ $f = e^{2\pi i\xi\cdot a}\underset{\underset{SO(d)\text{ mapping } e_d \text{ to } \omega_0}{\Uparrow}}{T} g$ would be supported on $\{\xi\in S^{d-1} : 1-\xi\cdot\omega_0 < \delta^2\}$

and $|(f d\sigma)^\wedge(x)| \gtrsim \delta^{d-1}$ on the tube ~~$\{x\in\mathbb{R}^d : |(x-a)\cdot\omega_0| < \frac{c^2}{\delta^2},\ |x-((x-a)\cdot\omega_0)\omega_0| < \frac{c}{\delta}$~~

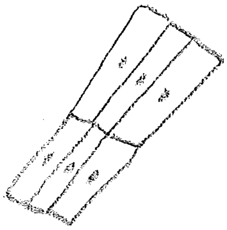
$$T^{a,\omega}_{c/\delta} = \{x\in\mathbb{R}^d : |(x-a)\omega_0| < \frac{c^2}{\delta^2},\ (x-a)-((x-a)\omega_0)\omega_0| < \frac{c}{\delta}\}$$



$(f d\sigma)^\wedge(x)$ is called a wave packet adapted to $T^{a,\omega}_{c/\delta}$

Correspondingly, we can construct linear combinations of Knapp examples and effectively end up with a Fourier series

I.e., ~~let~~ suppose f_ω is supported on $K_\delta^\omega = \{\xi \in S^{d-1} : 1 - \xi \cdot \omega < \delta^2\}$ in Fourier space and let us tile $\mathbb{R}^d$ into dual tubes $(T_{\omega\delta}^{a,\omega})_{a\in\mathbb{R}^d}$ where $a \in \mathbb{R}^d$ are of



the form $a_j = \begin{pmatrix} j' \cdot \frac{c}{\delta} \\ j_d \cdot \frac{c^2}{\delta^2} \end{pmatrix}$ $\quad j = (j', j_d) \in \mathbb{Z}^d$ if $\omega = e_d$

or generally $R_\omega a_j$ where $\underset{SO(d)}{\underset{\cap}{R_\omega}} e_d = \omega$

The collection of those a's is called $\Pi_{\omega,\delta}$

they're at most $\mathcal{O}(1)$ overlapping!

→ build up $\sum_{a \in \Pi_{\omega,\delta}} \underbrace{f_\omega^a}_{\|f_\omega^a\|_\infty \sim 1} e^{2\pi i \xi \cdot a}$ where $\operatorname{supp} f_\omega^a \subseteq K_\delta^\omega$

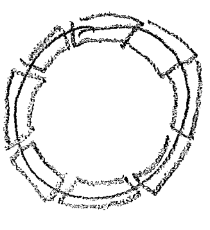
→ finally we could also sum over all frequencies $\omega \in S^{d-1}$ that are δ-separated, there are roughly $\delta^{-(d-1)}$ of them and shall be collected in Ω_δ

→ a general fct. f on S^{d-1} can be decomposed

$$f = \sum_{\omega \in \Omega_\delta} \sum_{a \in \Pi_{\omega,\delta}} f_\omega^a e^{2\pi i \xi \cdot a}$$

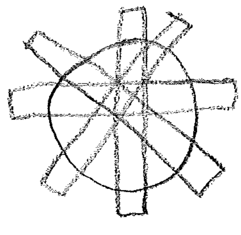
then $(f d\sigma)^\wedge$ gets decomposed analogously → this is called wave packet decomposition.

It will be interesting to observe what will happen if we forget about different modulations and randomize the coefficients $(f_\omega)_{\omega \in \Omega_\delta}$. We will get a bunch of dual tubes all running through the origin and (negatively) interferring with each other. ~~The~~ The randomization eventually allows ~~them~~ us to compute $\|\sum f_{T_\omega}\|_{L^4}$ and we shall find that it is surprisingly small meaning that tubes overlap quite a bit.



frequency space

→ collection of real-valued, unmodulated Knapp examples



physical space

(geometrically: it's possible to arrange the tubes in such a way that the compression

$$|\bigcup_T \tilde{T}| \le \frac{1}{K} \sum_T |T| , \quad K \sim \frac{\log 1/\delta}{\log\log 1/\delta}$$

$\tilde{T}$ .. shift of T by $2\delta^{-2}$ units in direction of long side

Since we have already gotten so far, let us actually go through the wp decomposition.

Now suppose we fixed the frequency and decomposed $f_\omega(\xi) = \sum_{a\in\Pi_{\omega,\delta}} f_\omega^a e^{2\pi i\xi\cdot a}\chi_\omega(\xi)$ ← now numbers!

where ~~$f_\omega^a = \langle f_\omega, e^{-2\pi}$~~ $f_\omega^a = \langle e^{-2\pi i\langle a,\cdot\rangle}, f_\omega\rangle$ $\quad |f_\omega^a| \sim \delta^{-(d-1)}$

Then $F_S^*\left(e^{2\pi i a\cdot\xi}\chi_\omega\right)$ ~~$\frac{1}{|T^{a,\omega}_{c/\delta}|}$~~ $\equiv \frac{1}{|T_{c/\delta}^{a,\omega}|} e^{2\pi i x\cdot\xi_\omega}\cdot\varphi_{T^{a,\omega}_{c/\delta}}(x)$ where ξ_ω is the center of the (frequency) support of f_ω and $\varphi_{T^{a,\omega}_{c/\delta}} \in \mathcal{S}$ is adapted to $T_{c/\delta}^{a,\omega}$

$\equiv \psi_{T^{a,\omega}_{c/\delta}}(x)$

has L^∞-norm $\sim |T^{a,\omega}_{c/\delta}|\cdot\delta^{d-1}$ $\sim \delta^{-2}$

$\Rightarrow F_S^* f_\omega = \sum_{a\in\Pi_{\omega,\delta}} f_\omega^a \psi_{T^{a,\omega}_{c/\delta}}(x)$ ← has L^∞ norm $\sim \delta^{d-1}$

and $\|f_\omega\|_{L^2(S)} \sim \left(\sum_a |f_\omega^a|^2\right)^{1/2} \delta^{\frac{d-1}{2}}$ $\Leftrightarrow$ $\|f_\omega\|_{L^2_{avg}(\kappa_{\delta,\omega})} \sim \left(\sum_a |f_\omega^a|^2\right)^{1/2}$

Plancherel $\qquad$ from $\|\chi_\omega\|_{L^2(S)}$

$\left(\|g\|_{L^p_{avg}(\Omega)} = |\Omega|^{-1/p}\|g\|_{L^p(\Omega)}\right)$ $\qquad$ and by almost orthogonality $\|f\|_{L^2(S)} \sim \delta^{\frac{d-1}{2}}\left(\sum_{\substack{\omega\in\Omega_\delta \\ a\in\Pi_{\delta,\omega}}} |f_\omega^a|^2\right)^{\frac{1}{2}}$

$\|\varphi^{a,\omega}_{T_{c/\delta}}\|_{L^2} \sim \delta^{-2}\cdot\delta^{-(d+1)/2}$ $\Rightarrow \delta^{d+1-2-\frac{d+1}{2}}$

$|T_{c/\delta}| \sim \delta^{-d-1}$ $\qquad = \delta^{d/2-1/2}$

4.2 Tomas-Stein theorem

As far as affirmative answers to the restriction conjecture go, we note that the case $d=2$ and $q=2$, $d\geq 3$ is settled. The latter goes back to Stein and Tomas.

Thm 4.6 $\quad F_S : L^p(\mathbb{R}^d) \to L^2(S)$ boundedly for ~~ap~~ all $p\in\left[1, \frac{2d}{d+3}\right]$ whenever S is a smooth compact codimension one manifold with nowhere vanishing curvature

Pf (Non-endpoint version) By T^*T, we have $F_S^*F_S f = \widehat{d\sigma} * f$.

Since $|\widehat{d\sigma}(x)| \sim \langle x\rangle^{-(d-1)/2}$, one already gets $F_S^*F_S : L^p \to L^{p'}$ for

$$\underbrace{1+\frac{1}{p'}-\frac{1}{p}}_{\frac{2}{p'}} = \frac{1}{r} \quad\text{and}\quad r\in\left[\frac{2d}{d-1},\infty\right] \quad \text{i.e., } p' \geq \frac{4d}{d-1} \quad \text{resp. } p \leq \frac{4d}{3d+1}$$

by the weak Young inequality $\left(\|\varphi * \psi\|_{p'} \leq \|\varphi\|_p \|\psi\|_{L^{r,\infty}}\right.$ when $1+\frac{1}{p'} = \frac{1}{p}+\frac{1}{r}$ and $1 < p,r \leq p' < \infty$

We can do vastly better by chopping $\widehat{d\sigma}$ up into ~~sx~~ chunks where the chopping is such that the pieces far out will have less impact.

$$\widehat{d\sigma} * f = \sum_{k\geq 0} \widehat{d\sigma}\, \psi_k * f \qquad \text{where} \qquad \psi_k(x) = \begin{cases} 1 & \text{if } 2^{k-1} < |x| < 2^k \\ 0 & \text{else (but smoothly)} \end{cases}$$

$$\equiv \sum_{k\geq 0} T_k * f \qquad\qquad \psi_0(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 2 \end{cases}$$

We shall now obtain $L^2 \to L^2$ and $L^1 \to L^\infty$ bounds for the T_k, interpolate between them and sum them up

$k=0$: $\widehat{d\sigma}\, \psi_0 \in C_c^\infty$ and hence $\| \widehat{d\sigma}\, \psi_0 * f \|_{p'} \leq \|f\|_p \, \|\widehat{d\sigma}\, \psi_0\|_r$ easily

$k \geq 1$ $\quad \|T_k f\|_{L^\infty} \leq \|f\|_1 \, \|\widehat{d\sigma}\, \psi_k\|_{L^\infty} \lesssim \|f\|_1 \, 2^{-k(d-1)/2}$ by $|\widehat{d\sigma}(x)| \sim \langle x \rangle^{-(d-1)/2}$

$$\|T_k f\|_2 = \|f\|_2 \, \| \widehat{\psi}_k * d\sigma \|_{L^\infty} \; ; \qquad \widehat{\psi}_k(\xi) \sim 2^{kd} \psi(2^k \xi) \, \cancel{\exp}(2\pi i \cdot 2^k \xi)$$

$$\lesssim 2^k \|f\|_2 \qquad\qquad \sup_{\xi \in S} \int_S \widehat{\psi}_k(\xi - \eta)\, d\sigma_S(\eta) \sim 2^k$$

$\Rightarrow \quad \|T_k\|_{p \to p'} \lesssim 2^{-k\frac{(d-1)}{2}\cdot\theta + k(1-\theta)}$ with ~~$\frac{1}{p} = \theta$, i.e., $1-\theta = \frac{1}{p'}$~~ $\quad \frac{1}{p} = \theta + \frac{1-\theta}{2} = \frac{1+\theta}{2}$

$$\theta = \frac{2}{p} - 1 = \frac{1}{p} - \frac{1}{p'} = \frac{1}{q}$$

$$k\frac{(d-1)}{2}\theta > k(1-\theta) \Leftrightarrow \frac{1}{2}(d-1)\theta + \theta > 1 \Leftrightarrow \theta > \frac{2}{d+1}$$

$$q < \frac{d+1}{2} \Leftrightarrow p < \frac{2(d+1)}{d+3}, \quad p' > \frac{2(d+1)}{d-1} \quad \blacksquare$$

for non-endpoint

With the same strategy one can prove Hölder continuity of TS for $\mathbb{S}^{d-1}$, si

<u>Thm 4.7</u> Let $S_t = \{ \xi \in \mathbb{R}^d : |\xi| = 1 + t \}$ for $|t| < 1$ and $S_0 = \mathbb{S}^{d-1}$.

Let $1 \leq p < \frac{2(d+1)}{d+3}$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{p'}$ i.e., $q \in [1, \frac{d+1}{2})$.

If $\alpha \in (0, \min\{\frac{d+1}{2} - q, q\})$, then

$$\sup_{|t|<1} \| \mathcal{F}_{S_t}^* \mathcal{F}_{S_t} - \mathcal{F}_S^* \mathcal{F}_S \|_{p \to p'} \lesssim_{\alpha q \varepsilon} |t|^{\alpha/q}.$$

Pf HW

To get the endpoint, recall (Stein-Shakarchi IV, 8.4.4)

Thm 4.8 (Stein interpolation)

Let T_z be an operator acting on functions on $\mathbb{R}^d$ that depends holomorphically on z in the strip $S=\{z\in\mathbb{C}: a<\mathrm{Re}(z)<b\}$ s.t. the form $(g,T_zf)_{L^2(\mathbb{R}^d)}$ converges absolutely, is holomorphic in S, continuous on ∂S with ~~at most~~ growth at most $\exp(O_{T,f,g}(\exp((\pi-\delta)|z|)))$ for some $\delta>0$. (as in 3-lines-lemma)

Let $0<p_0,p_1\leq\infty$ and $0<q_0,q_1\leq\infty$, $A_0,A_1>0$ s.t.

$$\|T_{z_j}f\|_{L^{q_j}}\leq A_j\|f\|_{L^{p_j}} \quad \cancel{\mathrm{Re}\,z=j} \quad j=0,1 \qquad z_0=a,\ z_1=b$$

Then $\|T_zf\|_{L^{q_\theta}}\leq A_0^{1-\theta}A_1^{\theta}\|f\|_{p_\theta}$ whenever $\mathrm{Re}\,z\in[a,b]$ and $\mathrm{Re}\,z=(1-\theta)a+\theta b$

Pf of Thm 4.6 (endpoint case)

Recall that ~~so~~ it suffices to show that $K(x)=\int_S d\sigma(\xi)\,e^{2\pi ix\cdot\xi}$ is an $L^p\to L^{p'}$-bdd convolution operator for $p=\frac{2(d+1)}{d+3}$ ($p<\ldots$ follow from Riesz-Thorin)

To that end, we embed this kernel in ~~the family (complex)~~ a complex family.

$$F_S^*F_S\underset{\substack{\shortparallel\\ \check{g}\\ \nwarrow\\ C^\infty}}{\overset{f_c\text{ (compactly supp)}}{f}}=\int_S\underbrace{d\sigma(\xi)\,\psi(\xi)}_{=d\mu(\xi)}e^{2\pi ix\cdot\xi}g(\xi).$$

Let us decompose S into so small pieces that each piece can, when appropriately translated and rotated, be described by the graph of a smooth fct φ, i.e., $\xi_d=\varphi(\xi')$, with $\varphi(0)=(\nabla_{\xi'}\varphi)(\xi'=0)=0$.

Recall that the Lebesgue measure on this piece is given by $d\sigma(\xi')=\sqrt{1+|\nabla\varphi(\xi')|^2}\,d\xi'$

$$\to F_S^*F_S f=\int_{\mathbb{R}^{d-1}}d\xi'\sqrt{1+|\nabla\varphi(\xi')|^2}\underset{\psi(\xi',\varphi(\xi'))}{\wedge}\exp(2\pi i(x'\cdot\xi'+x_d\varphi(\xi')))\,g(\xi',\varphi(\xi'))$$

Now consider $K_s(\xi)=\gamma_s(\xi_d-\varphi(\xi'))_+^{s-1}\psi_0(\xi)$ with $\psi_0(\xi)=\psi(\xi)\sqrt{1+|\nabla\varphi(\xi')|^2}$

$$\gamma_s=\underbrace{s(s+1)\ldots(s+N)}\underbrace{e^{s^2}}_{\substack{\text{will mitigate growth of}\\ s(s+1)\ldots(s+N)\text{ as }\mathrm{Im}\,s\to\infty}} \qquad N\geq\frac{d-1}{2}\text{ fixed}$$

We need the following properties about $K_s(\xi) = \gamma_s \psi_0(\xi)\,(\xi_d - \varphi(\xi'))_+^{s-1}$

Prop 4.9 The FT $\widehat{K}_s(x)$ is analytically continuable into the half-plane $-\frac{d-1}{2} \leq \mathrm{Re}(s)$ and satisfies $\sup_{x \in \mathbb{R}^d} |\widehat{K}_s(x)| \leq M$ in the strip $-\frac{d-1}{2} \leq \mathrm{Re}(s) \leq 1$

The proof is based on the following one-dim FT calculation. ~~Suppose F is~~ Suppose $F \in C_c^\infty(\mathbb{R})$ and let $I_s(p) = s(s+1)..(s+N) \int_0^\infty u^{s-1} F(u) e^{-2\pi i p u}\, du$, $p \in \mathbb{R}$

Lemma 4.10 ~~A~~ The fct $I_s(p)$ initially given above for $\mathrm{Re}(s) > 0$ has an analytic continuation into the half-space $\mathrm{Re}(s) > -N-1$. Moreover

(1) $|I_s(p)| \leq a_s (1+|p|)^{-\mathrm{Re}(s)}$ when $-N-1 < \mathrm{Re}(s) \leq 1$

(2) $I_0(p) = N!\, F(0)$

Here a_s is at most polynomially growing in $\mathrm{Im}(s)$ and depends only on the C^{N+1}-norm of F and its support.

Note that when $p=0$, we're dealing with the analytic continuation of the homogeneous distribution x_+^{s-1}

Pf of Lemma 4.10

Write $s(s+1)..(s+N) u^{s-1} = \left(\frac{d}{du}\right)^{N+1} u^{s+N}$ $\Rightarrow$ by integration by parts (N times),

$I_s(p) = (-1)^{N+1} \int_0^\infty u^{s+N} \left(\frac{d}{du}\right)^{N+1} \left(F(u) e^{-2\pi i p u}\right) du$ from which the analytic continuation to $\mathrm{Re}(s) > -N-1$ is evident. Moreover (1) follows whenever p is bounded, like when $|p| \leq 1$.

The proof of (1) is similar but requires more care when $|p| > 1$. So study the integral in $I_s(p) = s(s+1)..(s+N) \int_0^\infty u^{s-1} F(u) e^{-2\pi i p u}\, du$ in ranges $u|p| \lessgtr 1$.

Suppose $\eta \in C_c^\infty(\mathbb{R})$ bump located at origin.

a) Consider $u|p| < 1$: $\left| \int_0^\infty u^{s+N} \left(\frac{d}{du}\right)^{N+1} \left(F(u)\, \eta(u|p|)\, e^{-2\pi i p u}\right) du \right|$

$$\lesssim_F (1+|p|)^{N+1} \int_{0 \leq u < |p|^{-1}} u^{\sigma+N}\, du \quad \text{with } \sigma = \mathrm{Re}(s)$$

$$\overset{\sigma+N>-1}{\lesssim} (1+|p|)^{N+1} |p|^{-\sigma-N-1} \leq (1+|p|)^{-\sigma} \quad (|p|>1 \text{ here})$$

~~b) $u|p| \geq 1$ $\int_0^\infty$~~

b) $u|\rho|>1$: $(-1)^{N+1}\int_0^\infty u^{s+N}\left(\frac{d}{du}\right)^{N+1}\left((1-\eta(\rho u))F(u)e^{-2\pi i\rho u}\right)du$

$$= s(s+1)\dots(s+N)\,\frac{1}{(-2\pi i\rho)^k}\int_0^\infty u^{s-1}F(u)(1-\eta(\rho u))\left(\frac{d}{du}\right)^k\left(e^{-2\pi i\rho u}\right)du$$

where k is chosen so that $\mathrm{Re}\, s<k$. Then up to a factor that does not depend on ρ and is a polynomial in s, the integral equals

$$\rho^{-k}\int_0^\infty e^{-2\pi i\rho u}\left(\frac{d}{du}\right)^k\left[u^{s-1}F(u)(1-\eta(\rho u))\right]du$$

Since F has support in some interval $|u|\le A$, ~~it~~ the above integral is bdd by const $\rho^{-k}\int_{1/(2\rho)}^{A}du\; u^{\sigma-k-1}=\mathcal{O}(\rho^{-\sigma})$ because $\sigma=\mathrm{Re}(s)<k$

Combining analysis in a) and b) shows the claimed ~~the~~ bound in (1).

Besides the integration by parts also showed that

$$I_s(\rho)=-\cancel{s}(s+1)\dots(s+N)\int_0^\infty du\; u^s\frac{d}{du}\left(F(u)e^{-2\pi i\rho u}\right)du \;=\; \cancel{s}(s+1)\dots(s+N)\int_0^\infty \underbrace{su^{s-1}}_{d_u u^s}F(u)e^{-2\pi i\rho u}du$$

IBP

Setting $s=0$ gives ② since $F(0)=-\int_0^\infty\frac{d}{du}\left(F(u)e^{-2\pi i\rho u}\right)du$ $\square$

Pf of Prop 4.9 $\quad K_s^{(\varphi)}=\gamma_s\left(\xi_d-\varphi(\xi')\right)_+^{s-1}\psi_0(\xi) \qquad \gamma_s=s(s+1)\dots(s+N)e^{s^2}$

When $\mathrm{Re}(s)>0$, the change of variables $u=\xi_d-\varphi(\xi')$ yields

$$\widehat{K}_s(x)=\gamma_s\int_{\mathbb{R}^d}\left(\xi_d-\varphi(\xi')\right)_+^{s-1}\psi_0(\xi)\,e^{-2\pi i(x'\cdot\xi'+x_d\xi_d)}\,d\xi$$

$$=\gamma_s\int_0^\infty du\; u^{s-1}e^{-2\pi iu\cancel{x}_d}\int_{\mathbb{R}^{d-1}}d\xi'\; e^{-2\pi i(x'\cdot\xi'+x_d\varphi(\xi'))}\,\psi_0(\xi',\varphi(\xi')+u)$$

$$=e^{s^2}I_s(x_d) \qquad \left(\text{recall } I_s(\rho)=s(s+1)\dots(s+N)\int_0^\infty du\; u^{s-1}F(u)e^{-2\pi iu\rho}\right)$$

with $F(u)=\int_{\mathbb{R}^{d-1}}d\xi'\; e^{-2\pi i(x'\cdot\xi'+x_d\varphi(\xi'))}\,\psi_0(\xi',\varphi(\xi')+u)$

but by the usual stationary phase machinery for FT of surface measures, we have

$$|F(u)|\lesssim(1+|\cancel{x}|)^{-(d-1)/2} \quad \text{and} \quad |(D_u^\alpha F)(u)|\lesssim(1+|x|)^{-(d-1)/2} \quad \forall\,\alpha\in\mathbb{N}_0^d$$

But then by Lemma 4.10,

$$\widehat{K}_s(x)\le\underbrace{a_s}_{\substack{\text{at most polynomially}\\ \text{growing in } \mathrm{Im}(s)}}\;\underbrace{|e^{s^2}|}_{\le c\exp(-\mathrm{Im}(s)^2)\text{ when } -\frac{d-1}{2}\le\mathrm{Re}(s)\le 1}\,(1+|\cancel{x}_d|)^{-\mathrm{Re}(s)}\,(1+|x|)^{-(d-1)/2}$$

$\square$

Now back to

Pf of Thm 4.6 (endpoint case)

$$(F_s^* F_s f)(x) = \int d\xi' \, \Psi_0(\xi', \ell(\xi')) \, e^{2\pi i (x'\cdot\xi' + x_d \ell(\xi'))} g(\xi', \ell(\xi')) \qquad d\mu = d\sigma(\xi)\,\Psi(\xi)$$

$$K_s(\xi) = \gamma_s \, (\xi_d - \ell(\xi'))_+^{s-1} \Psi_0(\xi) \quad \text{with } \gamma_s = s(s+1)\dots(s+N)e^{s^2}$$

$$\hat{K}_0(x) = N! \, \widehat{d\mu}(x) \quad \text{by} \quad \hat{K}_s(x) = e^{s^2} \underbrace{I_s(x_d)} \quad \text{and} \quad I_0(\rho) = N!\, F(0)$$

$$\Downarrow \qquad\qquad = s(s+1)\dots(s+N)\int_0^\infty u^{s-1} F(u)\, e^{-2\pi i u \rho}\, du$$

$$F_s^* F_s f(x) = (\hat{K}_0 * f)(x) \qquad \text{with } F(u) = \int_{\mathbb{R}^{d-1}} d\xi' \, \Psi_0(\xi', u + \ell(\xi'))\, e^{-2\pi i (x'\cdot\xi' + x_d \ell(\xi'))}$$

$$\rightsquigarrow F(0) = \int d\xi' \sqrt{1 + |\nabla \ell(\xi')|^2}\, \Psi(\xi', \ell(\xi'))\, e^{-2\pi i x'\cdot\xi' + x_d \ell(\xi')}$$

$$= \widehat{d\mu}(x)$$

Now when $\mathrm{Re}(s) = 1$, $K_s(\xi)$ is bdd and hence (by definition) $\|K_s * f\|_{L^2} \lesssim \|f\|_{L^2}$

$\mathrm{Re}(s) = -\frac{d-1}{2}$, $\hat{K}_s(x)$ is bdd by Prop 4.9 $\Rightarrow$ $\|K_s * f\|_{L^\infty} \lesssim \|f\|_{L^1}$

$\Rightarrow$ by Stein interpolation (which we may apply since $(g, K_s * f)$ is continuous and holo on the strip $s \in \{z \in \mathbb{C} : \mathrm{Re}\, z \in (-\frac{d-1}{2}, 1), \mathrm{Im}\, z \in \mathbb{R}\}$ whenever $f, g \in L^1$, say)
with $a = -\frac{d-1}{2}$, $b = 1$, $\mathrm{Re}\, z = 0$, which means $0 = (1-\theta)\frac{1-d}{2} + \theta \Leftrightarrow \theta(\frac{d}{2} + \frac{1}{2}) = \frac{d-1}{2}$

$\Leftrightarrow \theta = \frac{d-1}{d+1}$

Since $p_0 = 1$, $q_0 = \infty$ and $p_1 = q_1 = 2$ here, we have indeed $p_\theta = \frac{2(d+1)}{d+3}$ ▨

We will immediately upgrade this to a substantially ~~better~~ better bound.
By Hölder, $F_s^* F_s : L^p \to L^{p'}$ is equivalent to $\|W_1 F_s^* F_s W_2\|_{\mathcal{B}(L^2)} \lesssim \|W_1\|_{2q} \|W_2\|_{2q}$
whenever $q \in [1, \frac{d+1}{2}]$

We upgrade this to a Schatten bound (due to Frank-Sabin) using the following general observation

Prop 4.11 Let T_z be an analytic family of operators on $\mathbb{R}^d$ in Stein's interpolation sense. Suppose $-d_0 \le \mathrm{Re}\, z \le 0$ for some $d_0 > 1$ and

$$\|T_{iy}\|_{2,2} \le M_0 e^{a|y|} \quad \text{and} \quad \|T_{-d_0 + iy}\|_{1,\infty} \le M_1 e^{b|y|} \quad \forall y \in \mathbb{R}$$

$\Rightarrow$ for $W_1, W_2 \in L^{2d_0}(\mathbb{R}^d, \mathbb{C})$, we have $\|W_1 T_{-1} W_2\|_{\mathfrak{S}^{2d_0}(L^2(\mathbb{R}^d))} \le M_0^{1-\frac{1}{d_0}} M_1^{\frac{1}{d_0}} \|W_1\|_{L^{2d_0}(\mathbb{R}^d)}$
$\|W_2\|_{L^{2d_0}(\mathbb{R}^d)}$

Corollary 4.12 Let $q \in [1, \frac{d+1}{2}]$, $W_1, W_2 \in L^{2q}(\mathbb{R}^d)$,

S a smooth ~~and~~ compact codimension one submanifold embedded in $\mathbb{R}^d$ with everywhere non-vanishing gaussian curvature

$$\Rightarrow \|W_1 \mathcal{F}_S^* \mathcal{F}_S W_2\|_{\mathfrak{S}^{\frac{(d-1)q}{d-q}}(L^2(\mathbb{R}^d))} \lesssim \|W_1\|_{L^{2q}} \|W_2\|_{L^{2q}}$$

Pf Follows ~~between~~ from interpolation in Schatten ideals between

$$\|W_1 \mathcal{F}_S^* \mathcal{F}_S W_2\|_{\mathfrak{S}^1(L^2(\mathbb{R}^d))} \overset{CS}{\le} \underbrace{\|\mathcal{F}_S W_1\|_{\mathfrak{S}^2(L^2(\mathbb{R}^d), L^2(S))}}_{= \sqrt{\int d\sigma_S(\xi) \int dx\, |W_1(x)|^2} = \sqrt{\sigma_S(S)}\, \|W_1\|_{L^2}} \|\mathcal{F}_S W_2\|_{\mathfrak{S}^2(L^2(\mathbb{R}^d), L^2(S))}$$

$$= \sigma_S(S) \|W_1\|_2 \|W_2\|_2$$

and $\|W_1 \mathcal{F}_S^* \mathcal{F}_S W_2\|_{\mathfrak{S}^{\frac{2p}{2-p}}(L^2(\mathbb{R}^d))} \lesssim \|W_1\|_{L^{\frac{2p}{2-p}}} \|W_2\|_{L^{\frac{2p}{2-p}}}$, $\quad p \in [1, \frac{2(d+1)}{d+3}]$

the latter being a consequence of Prop 4.11 and the proof of Thm 4.6 in the non-endpoint case. ◻

Pf of Prop 4.11 Since $\|\underset{\overset{\|}{|W_1| e^{i\varphi_1}}}{W_1} T_{-1} W_2\|_{\mathfrak{S}^{2\lambda_0}} \le \|e^{i\varphi_1}\|_{L^2, L^2} \|e^{i\varphi_2}\|_{L^2, L^2} \||W_1| T_{-1} |W_2|\|_{\mathfrak{S}^{2\lambda_0}}$, we can assume wlog that $W_1, W_2 \ge 0$. Moreover, by density, we may assume W_1, W_2 to be simple fcts.

Set $S_z := W_1^{-z} T_z W_2^{-z}$ which is still an analytic family ~~of~~ in Stein's sense in the strip $-\lambda_0 \le \mathrm{Re}(z) \le 0$ with $S_{-1} = W_1 T_{-1} W_2$.

We use Stein interpolation in Schatten ideals with ~~cf.~~ Simon - Trace ideals Thm 2.9

$$\|S_{iy}\|_{L^2, L^2} \le \|W_1^{-iy}\|_{L^\infty} \|W_2^{-iy}\|_{L^\infty} \|T_{is}\|_{L^2, L^2} \le M_0 e^{a|y|} \quad \text{and}$$

$$\|S_{-\lambda_0 + iy}\|_{\mathfrak{S}^2}^2 \le \int dx\,dy\, |W_1(x)|^{2\lambda_0} |W_2(y)|^{2\lambda_0} \underbrace{|T_{-\lambda_0 + iy}(x,y)|^2}_{\text{bdd since } T_{-\lambda_0+iy} : L^1 \to L^\infty \text{ bdd}} \overset{M_1 e^{b|y|}}{\le} \|W_1\|_{L^{2\lambda_0}}^{2\lambda_0} \|W_2\|_{L^{2\lambda_0}}^{2\lambda_0}$$

(Dunford-Pettis thm, cf. Cycon-Froese-Kirsch-Simon)

$\Rightarrow$ by Stein interpolation in Schatten ideals the claim is established ◻

4.3 Random Tomas-Stein in $\ell^2(\mathbb{Z}^d)$, $(d=2)$

$$(\mathcal{F}\underset{\underset{\ell^1(\mathbb{Z}^d)}{\uparrow}}{f})(\xi) = \sum_{n\in\mathbb{Z}^d} f(n)\, e^{-2\pi i n\cdot\xi} \quad , \quad \xi \in \mathbb{T}^d \text{ with fundamental cell } (-\tfrac{1}{2},\tfrac{1}{2})^d$$

$$(\mathcal{F}^*\psi)(n) = \int_{(-\frac{1}{2},\frac{1}{2})^d} d\xi\; e^{2\pi i n\cdot\xi}\, \psi(n)$$

Suppose S is a smooth compact codimension $\overset{one}{\vee}$ submanifold embedded in $(-\frac{1}{2},\frac{1}{2})^d$ with everywhere non-vanishing gaussian curvature.

Then by stationary phase, $|(d\sigma_S)^\wedge(n)| \lesssim (1+|n|)^{-(d-1)/2}$ and hence by adapting the previous machinery (interpolation thms hold in general Lions-Peetre interpolation spaces), one has

<u>Thm 4.13</u> Let $d\sigma_S$ be Lebesgue measure on S and suppose $\mu \ll \sigma_S$ with $\frac{d\mu}{d\sigma_S} \in L^2(S, d\sigma_S)$.

Then $\|\hat{\mu}\|_{\ell^{\frac{2(d+1)}{d-1}}(\mathbb{Z}^d)} \lesssim \|\frac{d\mu}{d\sigma_S}\|_{L^2(S)}$

As before, $\|\mathcal{F}_S V \mathcal{F}_S^*\|_{\mathcal{L}(L^2(S))} \lesssim \|V\|_{\ell^{\frac{d+1}{2}}}$ by Hölder (duality).

Upshot: when V is randomized, then it is allowed to decay almost twice as slowly.

Let $V_\omega(n) = \omega_n |n|^{-\epsilon} v(n)$ with $v \in \ell^{d+1}(\mathbb{Z}^d)$ and $\{\omega_n\}_{n\in\mathbb{Z}^d}$ a sequence of independent subgaussian random variables (such as Rademacher or Gaussian).

<u>Thm 4.14</u> In above situation, we have $\mathbb{E}_\omega \|\mathcal{F}_S V_\omega \mathcal{F}_S^*\|_{\mathcal{L}(L^2(S))} \lesssim_{\epsilon,d} \|v\|_{\ell^{d+1}(\mathbb{Z}^d)}$ for d:

where $(\mathcal{F}_S^* g)(n) = \int d\sigma_S(\xi)\, e^{2\pi i x\cdot\xi} g(\xi)$, $(\mathcal{F}_S f)(\xi) = \sum e^{-2\pi i n\cdot\xi} f(n)\Big|_{\xi\in S}$.

This theorem is due to Bourgain (2002, 2003) who stated it for the level surfaces of the discrete Laplace that are curved iff and only if $d=2$.

His arguments however only need Tomas-Stein, i.e, curvature.

However, a generalization to $L^2(\mathbb{R}^d)$ seems to be more complicated.

We state his main tools for the proofs without proof and refer to R. Vershynin's recent textbook (2018) for details and further references.

Lemma 4.15 (Maximum of sub-gaussians)

Let $(X_j)_{j\in 1..N}$ be a sequence of independent random subgaussian variables.

Then $\mathbb{E}\max_{1\le j\le N}|X_j| \lesssim \sqrt{\log N}\max_{1\le j\le N}\underbrace{\|X_j\|_{\psi_2}}$

$= \inf\{t>0 : \mathbb{E}(\exp(\frac{X_j^2}{t^2}))\le 2\} < \infty :\Leftrightarrow X_j$ sub-gaussia

Pf Vershynin, Ch. 2

Lemma 4.16 (Sums of independent sub-gaussians)

Let $X_1 .. X_N$ be independent sub-gaussian random variables.

Then $\|\sum_{j=1}^N X_j\|_{\psi_2}^2 \le \sum_{j=1}^N \|X_j\|_{\psi_2}^2$

Pf Vershynin Ch. 2

(n = fixed → think of $\mathcal{E}$ as a finite point ~~set~~ $\mathbb{R}^n$)

Corollary 4.17 Let $\{\omega_m\}_{m\in\mathbb{N}}$ be independent sub-gaussian random variables and $\overset{\cup_1}{\mathcal{E}}$ be a separable (possibly infinite-dim.) vector space over $\mathbb{C}$ with cardinality $|\mathcal{E}|$.
i.e. $n=\infty$ allowed

$\mathcal{E} \ni \xi = (\xi_m)_{m\in\mathbb{N}} = (\xi_1, \xi_2, ..)$

Then $\cancel{\langle \sum_n \omega_n \xi_n \rangle}$

$$\mathbb{E}\sup_{\xi=(\xi_m)_m\in\mathcal{E}}\left|\sum_{m=1}^n \omega_m \xi_m\right| \lesssim \sqrt{\log|\mathcal{E}|}\sup_{\xi\in E}\sqrt{\sum_{m=1}^n |\xi_m|^2}$$

Pf Identify X_j ~~≠~~ with $\sum_{m=1}^n \omega_m \xi_m^{(j)}$ where $(\xi_m^{(j)})_{m\in\mathbb{N}}$ denote the elements of the vector $\xi^{(j)}\in\mathcal{E}$. There are at most $|\mathcal{E}|$ many random variables X_j then.

So $$\mathbb{E}\sup_{\xi\in\mathcal{E}}\left|\sum_{m=1}^n \omega_m\xi_m\right| \underset{4.15}{\lesssim} \sqrt{\log|\mathcal{E}|}\sup_{\xi\in\mathcal{E}}\underbrace{\|\sum_{m=1}^n \omega_m \xi_m\|_{\psi_2}}_{\underset{4.16}{\lesssim}\left(\sum_{m=1}^n|\xi_m|^2\right)^{1/2}}$$ ⊜

See also Exercise 7.5.10 in Vershynin: Let T be a finite set of points in $\mathbb{R}^n$, then

$$v(T) := \mathbb{E}\sup_{x\in T}\langle x, \sum^n \underset{\substack{\text{gaussian} \sim N(0,1)}}{g_j} e_j\rangle \lesssim \sqrt{\log\underbrace{|T|}_{\text{cardinality}}}\ \operatorname{diam} T$$

$e_j = (0, .. 1, 0 ..)$

Cor 4.17 ~~sometimes~~ goes under the name of Dudley's inequality.
(See also Ch. 8 in Vershynin, when the discrete family of random variables is replaced by random process with sub-gaussian increments)

The next tool comes from high-dimensional geometry of Banach spaces and is originally due to Pajor - ~~Tomczak~~ Tomczak - Jagermann

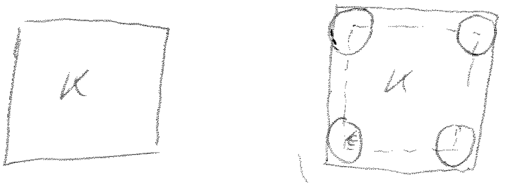
Consider the euclidean unit ball $B_n^2 = \{ x \in \mathbb{R}^n : \sum_{j=1}^{n} x_j^2 \leq 1 \}$ and let $\| \cdot \|_X$ be another (semi)norm on $\mathbb{R}^n$. Let $B_X = \{ x \in \mathbb{R}^n : \|x\|_X \leq 1 \}$.

Q: What's the minimal number of balls ϵB_X needed to cover B_n^2?

$$N(B_n^2, \| \cdot \|_X, \epsilon) := \min \{ N \in \mathbb{N} : \exists \, t_1 .. t_N \in \mathbb{R}^n : B_n^2 \subseteq \bigcup_{j=1}^{N} t_j + \epsilon B_X \}$$

Example If we take $\| \cdot \|_X$ to be euclidean norm again, then $\epsilon^{-n} \leq N(B_n^2, \| \cdot \|_2, \epsilon) \leq \left(1 + \frac{2}{\epsilon}\right)^n$

In general, if $K \subseteq \mathbb{R}^n$, then $\frac{|K|}{|\epsilon B_n^2|} \leq N(K, \| \cdot \|_2, \epsilon) \leq \frac{|(K + \frac{\epsilon}{2} B_2^n)|}{|\frac{\epsilon}{2} B_n^2|}$



$K + \frac{\epsilon}{2} B_2^n = \{ x + y : x \in K, \, y \in \frac{\epsilon}{2} B_2^n \}$

It's useful to have another interpretation in mind. $N(B_n^2, \| \cdot \|_X, \epsilon)$ is the minimal size of the discrete set $\mathcal{E} \subseteq \mathbb{R}^n$ (the ϵ-net) s.t.

$$\max_{x \in B_2^n} \min_{x' \in \mathcal{E}} \| x - x' \|_X < \epsilon$$

Thm 4.18 (dual to Sudakov) $\log N(B_n^2, \| \cdot \|_X, \epsilon) \lesssim n \cdot \epsilon^{-2} \cdot \underbrace{\left(\int_{S^{n-1}} \|x\|_X \, \overset{\substack{\text{normalized Haar} \\ \text{measure on } S^{d-1}}}{d\mu(x)} \right)^2}_{\equiv A_r \ \text{(average)}}$

Note that $A_r = \text{const} \; n^{-1/2} \int_{\mathbb{R}^n} dx \; \|x\|_X \exp\left(- \frac{\|x\|_2^2}{2} \right)$

$$= \text{Const} \; n^{-1/2} \int_{\mathbb{R}^n} \underbrace{\int_{\Omega}}_{\substack{\text{any probability} \\ \text{space w/ prob. measure } dp(\omega)}} dp(\omega) \left\| \sum_{j=1}^{n} \underbrace{g_j}_{\text{gaussian}}(\omega) e_j \right\|$$

For a short proof of Thm 4.18 see Bourgain - Lindenstrauss - Milman.

Pf of Thm 4.14

$$\underbrace{\mathbb{E}_\omega \| F_S V_\omega F_S^* \|}\lesssim_\epsilon \|v\|_{\ell^{d+1}(\mathbb{Z}^d)} \qquad V_\omega(n) = \sum_{\ell \geq 0} V_\ell \quad \text{where} \quad V_0 = V_\omega \mathbb{1}_{|n|<1}$$

$$V_\ell = V_\omega \mathbb{1}_{2^{\ell-1}<|n|<2^\ell}$$

$$A_\ell = \tilde{A}_\ell \cap \mathbb{Z}^d \text{ with } \tilde{A}_\ell = \{x \in \mathbb{R}^d : 2^{\ell-1} \leq |x| \leq 2^\ell\}$$

$$\leq \sum_{\ell \geq 0} \mathbb{E}_\omega \underbrace{\| F_S V_\ell F_S^* \|}$$

$$= \sup_{\mu_1, \mu_2 \in L^2(S, d\sigma_S)} |(\hat{\mu}_1, V_\ell \hat{\mu}_2)| = \sup_{\mu_1,\mu_2} 2^{-\epsilon\ell} \left| \sum_{n \in A_\ell} \hat{\mu}_1(n) \hat{\mu}_2(n) \omega_n v(n) \right|$$

(with $\|\frac{d\mu_i}{d\sigma}\|_{L^2(S)} \leq 1$)

The idea is to find a deterministic estimate (using classical Tomas-Stein) and a probabilistic estimate and optimize between them.

To that end, we shall decompose the $\hat{\mu}_j(n)$ in smaller bits, using dual to Sudakov.

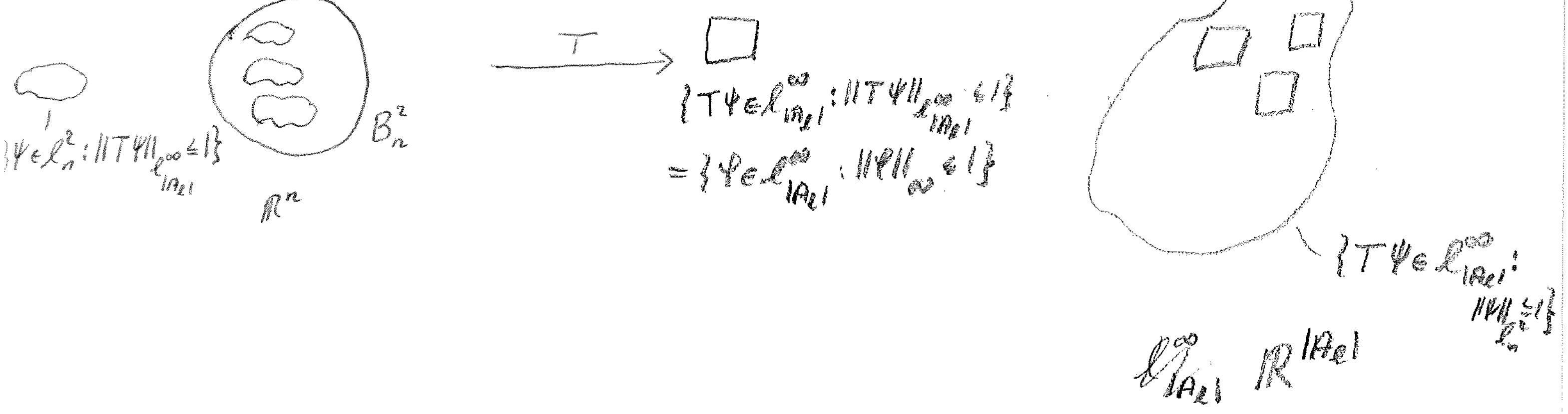
Suppose first that the ∞-dim space $L^2(S)$ is discretized in $\ell_n^2 = (\mathbb{R}^n : \|\cdot\|_2)$ and we will let $n \to \infty$ at the end of the argument.

By dual to Sudakov we could cover $\{\mu \in \ell_n^2 : \|\mu\|_2 \leq 1\}$ with balls in metric $\|\cdot\|_X$.

We now construct $\|\cdot\|_X$ as follows. Let $\underset{\overset{?}{F_S^*}}{T} : \ell_n^2 \to \ell^\infty(A_\ell)$ be a bounded operator, then $B_X \equiv \{\Psi \in \ell_n^2 : \|T\Psi\|_{\ell^\infty(A_\ell)} \leq 1\}$

$\|\Psi\|_X \equiv \|T\Psi\|_{\ell^\infty(A_\ell)}$. Note that $\ell^\infty(A_\ell) = \ell^\infty_{|A_\ell|} = (\mathbb{R}^{|A_\ell|} : \|\cdot\|_\infty)$. $\underset{\cup\!\vert}{\ell_n^2}$

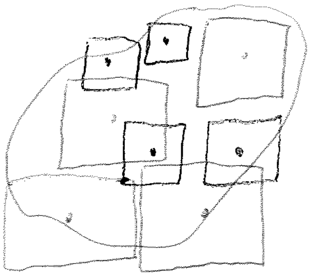
Now change perspectives and observe that covering $\{\mu \in \ell_n^2 : \|\mu\|_2 \leq 1\}$ with $\|T\cdot\|_{\ell^\infty(A_\ell)}$-balls is the same as covering $\{T\mu|_{\ell^\infty(A_\ell)} : \|\mu\|_{\ell_n^2} \leq 1\} \subseteq \ell^\infty(A_\ell)$ with balls in $\|\cdot\|_{\ell^\infty_{|A_\ell|}}$ metric.



So we cover now the set $\{T\mu|_{\ell^\infty(A_\ell)} : \|\mu\|_{\ell_n^2} \leq 1\}$ with ϵ-balls in $\ell^\infty_{|A_\ell|}$-metric. We collect this collection of balls into the set $\mathcal{E}_\epsilon \subseteq \ell^\infty_{|A_\ell|}$ and assume that it's the one with minimal cardinality. $\mathcal{E}_\epsilon$ consists of points (or vectors ξ) with the property

$$\max_{\mu \in \ell_n^2} \min_{\xi \in \mathcal{E}} \|\xi - T\mu\|_{\ell^\infty_{|A_\ell|}} < \epsilon$$

Now let us take $\epsilon = 2^{-r}$ with $r \in \mathbb{N}_0$, and consider two coverings $\mathcal{E}_{2^{-r-1}}$ and $\mathcal{E}_{2^{-r}} \in \ell^\infty_{|A_\ell|}$ of $\{T\mu|_{\ell^\infty_{|A_\ell|}} : \|\mu\|_{\ell^2_n} \leq 1\}$, the one net being twice as fine than the other.

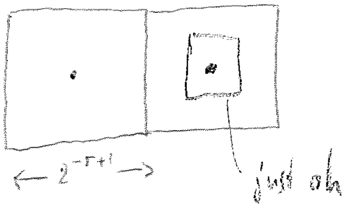


$\mathcal{E}_{2^{-r}}$ consists of blue centers

$\mathcal{E}_{2^{-r-1}}$ consists of black centers

$\mathcal{E}_{2^{-r-1}} - \mathcal{E}_{2^{-r}}$ has cardinality $|\mathcal{E}_{2^{-r-1}}| \cdot |\mathcal{E}_{2^{-r}}|$

$\Rightarrow$ choose an appropriate subset $\mathcal{F}_r \subseteq \mathcal{E}_{2^{-r-1}} - \mathcal{E}_{2^{-r}}$ such that for $\xi \in \mathcal{F}_r$, one has $\|\xi\|_{\ell^\infty_{|A_\ell|}} < 2^{-r+1}$, i.e., we consider only (centers of) black and blue cubes that are either contained within each other or so far distant from each other that their distance is less than the side length of the larger cube



$\Rightarrow$ we may decompose $T\mu|_{\ell^\infty_{|A_\ell|}}$ with $\|\mu\|_{\ell^2_n} \leq 1$ by considering a linear combination of finer and finer getting nets (maybe one of the centers coincides with $T\mu$; if not, decompose finer)

$$\Rightarrow \quad T\mu|_{\ell^\infty_{|A_\ell|}} = \sum_r \xi^{(r)} \quad \text{for some } \xi^{(r)} \in \mathcal{F}_r.$$

Before we continue we must make the connection between $L^2(S)$ and ℓ^2_n and estimate $|\mathcal{F}_r| \leq |\mathcal{E}_{2^{-r-1}}| \cdot |\mathcal{E}_{2^{-r}}|$ which will (as aspired) not depend on n. This will allow us to pass to $n \to \infty$ and understand T as the extension operator.

Recall dual-to-Sudakov asserting $\log N(\{T\mu|_{\ell^\infty_{|A_\ell|}} : \|\mu\|_{\ell^2_n} \leq 1\}, B^\infty_{|A_\ell|}, \epsilon)$

$$= \log N(\{\mu \in \ell^2_n : \|\mu\|_{\ell^2_n} \leq 1\}, \{\psi \in \ell^2_n : \|T\psi\|_{\ell^\infty_{|A_\ell|}} \leq 1\}, \epsilon)$$

$\|\psi\|_X = \|T\psi\|_{\ell^\infty_{|A_\ell|}}$ for $\psi \in \ell^2_n$

$$\lesssim n \epsilon^{-2} A_r^2 \quad \text{where } ~~A_r = \text{const}\, n^{-1/2} \int_\Omega dp(\omega)~~$$

$$\text{where } A_r = \text{const}\, n^{-1/2} \int_\Omega dp(\omega) \left\| \sum_{j=1}^n g_j(\omega) T e_j \right\|_{\ell^\infty_{|A_\ell|}}$$

$$= \frac{c}{\sqrt{n}} \mathbb{E}_\omega \sup_{1 \leq m \leq |A_\ell|} \left| \sum_{j=1}^n g_j(\omega) (Te_j)_m \right|$$

$$\underset{\text{Dudley}}{\overset{4.17}{\lesssim}} n^{-1/2} \sqrt{\log |A_\ell|} \sup_{1 \leq m \leq |A_\ell|} \left(\sum_{j=1}^n |(Te_j)_m|^2 \right)^{1/2}$$

$$= n^{-1/2} \sqrt{d\ell}\, \|T\|_{\ell^2_n \to \ell^\infty_{|A_\ell|}}.$$

$\Rightarrow \log N\left(\{T\mu|_{\ell^\infty_{|A_\ell|}} : \|\mu\|_{\ell^2_n} \le 1\}, B^\infty_{|A_\ell|}, \epsilon\right) \lesssim \epsilon^{-2} \cdot n \cdot n^{-1} \cdot d\ell \cdot \|T\|^2_{\ell^2_n \to \ell^\infty_{|A_\ell|}}$

$\Rightarrow$ we may pass to $n \to \infty$ and replace T by $F_S^* : L^2(S) \to \ell^\infty$!

$\Rightarrow \quad \hat{\mu}_j(n) = \sum_r \xi_j^{(r)}, \quad \xi_j^{(r)} \in \mathcal{F}_r$ with $\|\xi_j^{(r)}\|_{\ell^\infty(A_\ell)} \le 2^{-r+1}$

$(j=1,2)$ and $\|\xi_j^{(r)}\|_{\ell^{2(d+1)/(d-1)}(\mathbb{Z}^d)} < C$ by Tomas-Stein.

$$\Rightarrow \mathbb{E}_\omega \|F_S V_\omega F_S^*\| \le \sum_{\ell \ge 0} 2^{-\epsilon\ell} \sum_{r_1, r_2 \ge 0} \mathbb{E}_\omega \max_{\substack{\xi_1 \in \mathcal{F}_{r_1} \\ \xi_2 \in \mathcal{F}_{r_2}}} \left|\sum_{n \in A_\ell} \omega_n v(n) \xi_1(n) \xi_2(n)\right| \equiv (*)$$

We're now in the situation to derive a deterministic and a probabilistic bound on $|\ldots|$

Deterministic bound: $\max_{\xi_1,\xi_2} \left|\sum_{n \in A_\ell} \omega_n v(n) \xi_1(n) \xi_2(n)\right| < 2^{-r_1-r_2} \sum_{n \in A_\ell} v(n) \lesssim \|v\|_{\ell^{d+1}} 2^{-r_1-r_2} \cdot 2^{\frac{d^2}{d+1}\ell}$

behaves nicely in r_1, r_2 but terribly in ℓ.

Probabilistic bound: We again use Dudley's estimate using $|\mathcal{F}_r| < |\mathcal{E}_{2^{-r-1}}| |\mathcal{E}_{2^{-r}}|$ ~~$\lesssim_{d,S} \ell^2 2^{r+4r}$~~

i.e., $\sqrt{\log|\mathcal{F}_r|} < \sqrt{\log|\mathcal{E}_{2^{-r-1}}| + \log|\mathcal{E}_{2^{-r}}|} \lesssim_{d,S} \sqrt{\ell} \cdot 2^r$

~~$\mathbb{E}_\omega \max_{\substack{\xi_1 \in \mathcal{F}_{r_1} \\ \xi_2 \in \mathcal{F}_{r_2}}} \left|\sum_{n \in A_\ell} \omega_n v(n) \xi_1(n) \xi_2(n)\right| \lesssim \sqrt{\log|\mathcal{F}_{r_1}| + \log|\mathcal{F}_{r_2}|} \max_{\xi_1,\xi_2} \left(\sum v(n)^2 \xi_1(n)^2 \xi_2(n)^2\right)^{\frac{1}{2}} \lesssim \ell 2^{r_1+r_2}$~~

$$\Rightarrow \mathbb{E}_\omega \max_{\substack{\xi_1 \in \mathcal{F}_{r_1} \\ \xi_2 \in \mathcal{F}_{r_2}}} \left|\sum_{n \in A_\ell} \omega_n v(n) \xi_1(n) \xi_2(n)\right| \lesssim \sqrt{\log|\mathcal{F}_{r_1}| + \log|\mathcal{F}_{r_2}|} \max_{\xi_1,\xi_2} \left(\sum_{n \in A_\ell} v(n)^2 \xi_1(n)^2 \xi_2(n)^2\right)^{\frac{1}{2}}$$

$\|$

$\max_{\mathcal{F}_{r_1} \times \mathcal{F}_{r_2}}$

$$\underset{\text{Hölder}}{\lesssim} \sqrt{\ell} (2^{r_1} + 2^{r_2}) \|v\|_{d+1} \|\xi_1 \cdot \xi_2\|_{\frac{2(d+1)}{d-1}}$$

$$\lesssim \sqrt{\ell} (2^{r_1} + 2^{r_2}) \min\{2^{-r_1}, 2^{-r_2}\} \|v\|_{d+1} \sim \sqrt{\ell}$$

$$\Rightarrow \mathbb{E}_\omega \|F_S V_\omega F_S^*\| < (*) \lesssim \sum_{\ell \ge 0} 2^{-\epsilon\ell} \sum_{r_1, r_2 \ge 0} \min\left\{2^{-r_1-r_2} \cdot 2^{\frac{d^2}{d+1}\ell}, \sqrt{\ell} \cancel{(2^{r_1}+2^{r_2}) \min\{2^{-r_1}, 2^{-r_2}\}}\right\} \|v\|_{d+1}$$

$$\lesssim \|v\|_{d+1} \sum_{\ell \ge 0} 2^{-\epsilon\ell} \left(\sum_{r_1+r_2 < \ell \cdot \frac{d^2+8}{d+1}} \sqrt{\ell} + \sum_{r_1+r_2 > \ell \cdot \frac{d^2+8}{d+1}} 2^{-(r_1+r_2)} 2^{\frac{d^2}{d+1}\ell}\right)$$

$$\sim \|v\|_{d+1} \sum_{\ell \ge 0} 2^{-\epsilon\ell} \left(\ell^{5/2} + 1 2^{-(\frac{d^2}{d+1}-2)\ell}\right) \sim_\epsilon \|v\|_{d+1} \text{ ~~if } d=2\text{~~} \quad ▨$$

(same issue as with restriction in $d \ge 3$; here, in $d=2$ we used bilinear approach

$d \ge 3$ multilinear approach necessary?

Surprisingly, a quite asymmetric version seems to hold in $d \geq 2$

Let $0 < \delta < 1$ s.t. $\delta < \frac{2\epsilon}{\frac{d}{d+1} + 2\epsilon}$ $\left(\to \frac{2\epsilon}{2\epsilon+1} \text{ if } d \to \infty \right)$ and let $V_\omega^\delta(n) = \omega_n v(n)^\delta 2^{-\epsilon\delta}$

$$|V_\omega|^\delta(n) = v(n)^\delta 2^{-\epsilon\delta}.$$

$$\Rightarrow \mathbb{E}_\omega \| V_\omega^\delta F_S^* F_S |V_\omega|^{1-\delta} \| \lesssim_{\epsilon,\delta,d} \|v\|_{d+1} \quad \text{in } \ell^2(\mathbb{Z}^d).$$

(doubt that since if $\underset{\shortparallel}{AB}$ normal and $\overset{\shortparallel}{BA}$ not, then $\|AB\| \leq \|BA\|$

~~Related to~~ $\quad 4\pi\, F_S V_\omega F_S^* \qquad V_\omega^\delta F_S^* F_S |V_\omega|^{1-\delta}$

4.4 Uniform resolvent estimates and connection to Tomas-Stein

Recall $\mathrm{Im} \frac{1}{\cancel{x} - \cancel{\lambda} i\epsilon} = \frac{\epsilon}{\epsilon^2 + (x-\lambda)^2}$ $\equiv$ Poisson kernel, i.e., an approximation of the identity.

Consider $(\psi, (-\Delta - z)^{-1}\psi) = \int d\xi\, |\hat{\psi}(\xi)|^2 (\xi^2 - z)^{-1} = \int_0^\infty dh\, \frac{h^{d-1}}{h^2 - z} \int_{S^{d-1}} d\omega\, |\hat{\psi}(h\omega)|^2$

$$= \frac{1}{2} \int_0^\infty dh\, \frac{h^{\frac{d}{2}-1}}{h - z} \int d\omega\, |\hat{\psi}(\sqrt{h}\omega)|^2$$

and its imaginary part $\mathrm{Im}(\psi, (-\Delta - \underset{\underset{\lambda + i\epsilon}{\shortparallel}}{z})^{-1}\psi) = \frac{1}{2} \int_0^\infty dh\, h^{\frac{d}{2}-1} \frac{\epsilon}{\epsilon^2 + (h-\lambda)^2} \underbrace{\int_{S^{d-1}} d\omega\, |\hat{\psi}(\sqrt{h}\omega)|^2}_{= h^{-\frac{d-1}{2}} (\psi, F_{\sqrt{h}S}^* F_{\sqrt{h}S} \psi)}$

$$= \frac{1}{2} \int dh\, h^{-\frac{1}{2}} \frac{\epsilon}{\epsilon^2 + (h-\lambda)^2} (\psi, F_{\sqrt{h}S^{d-1}}^* F_{\sqrt{h}S^{d-1}} \psi)$$

$$\xrightarrow{\epsilon \to 0} \frac{\pi}{2} \lambda^{-1/2} (\psi, F_{\sqrt{\lambda}S^{d-1}}^* F_{\sqrt{\lambda}S^{d-1}} \psi) = \mathrm{Im}(\psi, (-\Delta - \lambda - i0)^{-1}\psi)$$

Integrating this over $\lambda \in \Lambda \subseteq \mathbb{R}_+$ restores Stone's formula

$$(\psi, E_{-\Delta}(\Lambda)\psi) = \frac{1}{\pi} \int_\Lambda d\lambda\, \mathrm{Im}(\psi, (-\Delta - \lambda - i0)^{-1}\psi) = \frac{\pi}{\cancel{2}} \frac{1}{2} \int_\Lambda d\lambda\, \lambda^{-1/2} (\psi, F_{\sqrt{\lambda}S^{d-1}}^* F_{\sqrt{\lambda}S^{d-1}} \psi)$$

$\Rightarrow$ TS gives spectral cluster bounds, i.e., $L^p \to L^{p'}$-bounds on the spectral projection $E_{-\Delta}(\Lambda)$ (or any other operator $F(D)$ by a change of variables), namely

$$\int_\Lambda d\lambda\, \lambda^{-1/2} (\psi, F_{\sqrt{\lambda}S}^* F_{\sqrt{\lambda}S} \psi) \leq \int_\Lambda d\lambda\, \lambda^{-1/2 + \frac{d-1}{2}} \underbrace{\| F_S\, \psi(\tfrac{\cdot}{\sqrt{\lambda}}) \|_{L^2(S)}^2}_{\lesssim \| \psi(\frac{\cdot}{\sqrt{\lambda}}) \|_{L^p}^2 = \lambda^{\cancel{2}d/p}} \cdot \lambda^{-d}$$

$$\leq \int_\Lambda d\lambda\, \lambda^{\frac{d}{2} - 1 - d + d/p} \|\psi\|_p^2 = \int_\Lambda d\lambda\, \lambda^{-1 + \frac{d}{2}(\frac{1}{p} - \frac{1}{p'})} \|\psi\|_p^2$$

$$\Rightarrow \| d E_{\Delta}(\Lambda) \|_{p \to p'} \lesssim \int_\Lambda d\lambda\, \lambda^{-1 + \frac{d}{2}(\frac{1}{p} - \frac{1}{p'})} \qquad 1 \leq p \leq \frac{2(d+1)}{d+3}$$

$\rightsquigarrow \frac{1}{2}\left(\frac{2}{p} - 1\right) = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{p'}\right)$

$\rightarrow$ valuable e.g. for $-\Delta_g$ on compact (smooth) boundaryless Riemannian manifolds
$\hookrightarrow$ discrete spec; "spectral cluster estimate" deserves its name!

Hence, TS shows that imaginary parts of resolvent on its spectrum is in fact $L^p \to L^{p'}$ - bdd with $\|\operatorname{Im}(-\Delta-\lambda-i0)^{-1}\|_{p,p'} \lesssim \lambda^{-1+\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}$, $1 \le p \le \frac{2(d+1)}{d+3}$

~~Th~~ On the other hand, the celebrated Kenig-Ruiz-Sogge uniform resolvent estimate $\|(-\Delta-z)^{-1}\|_{p,p'} \lesssim |z|^{-1+\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}$ for $p \in [\frac{2d}{d+2}, \frac{2(d+1)}{d+3}]$ clearly implies TS. Let's prove this bound. Its proof again relies on complex Stein interpolation, hence we shall the upgrade it to a Schatten bound afterwards.

Thm 4.19 (KRS) Let $d \geq 3$, $p \in [\frac{2d}{d+2}, \frac{2(d+1)}{d+3}]$, (so that $\frac{1}{p}-\frac{1}{p'} \in [\underset{\uparrow\ \text{TS-exponent}}{\frac{2}{d+1}}, \underset{\uparrow\ \text{Sobolev exponent}}{\frac{2}{d}}]$)

$$\Rightarrow \|(-\Delta-z)^{-1}\|_{p,p'} \lesssim |z|^{-1+\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}$$

Rem The theorem also holds in $d=2$ (by inspection of its proof) if $1<p\le 6/5$

Pf Define $m_\lambda(\xi) = \frac{e^{\lambda^2}}{\Gamma(\lambda+d/2)}(\xi^2-z)^{+\lambda}$ which recovers the multiplier of the resolvent for $\lambda=-1$. Here, consider $\lambda \in \{z \in \mathbb{C} : \underset{\uparrow\ [-\frac{d+1}{2}, 0]}{\operatorname{Re}(\lambda)}\}$

$$\to \|m_{iy}(D)\|_{L^2,L^2} \lesssim 1 \quad \text{for } \begin{array}{l}\operatorname{Re}\lambda = 0\\ \operatorname{Im}\lambda = y\end{array}$$

Now for the $L^1 \to L^\infty$ bound let $\operatorname{Re}\lambda \in [-\frac{d+1}{2}, -\frac{d}{2}]$. ~~and~~ By explicit computations, see Gelfand-Shilov, $\hat{m}_\lambda(x) = \frac{e^{\lambda^2}2^{\lambda+1}}{\Gamma(-\lambda)\Gamma(\frac{d}{2}+\lambda)}\left(\frac{z}{x^2}\right)^{\frac{1}{2}(\lambda+d/2)} K_{\frac{d}{2}+\lambda}(\sqrt{z}|x|)$

- $|e^{\nu^2}\nu K_\nu(w)| \lesssim |w|^{-|\operatorname{Re}(\nu)|}$, $|w|\le 1$, $\operatorname{Re}(w)>0$
- $|K_\nu(w)| \lesssim e^{-\operatorname{Re}(w)}|w|^{-1/2}$, $|w|\geq 1$, $\operatorname{Re} w>0$, $\operatorname{Re}\nu \geq 0$

Setting $\lambda = -a+iy$, $\nu = \frac{d}{2}+\lambda = \frac{d}{2}-a+iy$, we have $\operatorname{Re}\nu \in [0, \frac{1}{2}]$ for $a \in [\frac{d}{2}, \frac{d+1}{2}]$

Thus, for $w=\sqrt{z}|x-y|$ with $z\neq 1$ but $|z|=1$, so that $|w|=|x-y|$, we can estimate

$$|K_\nu(w)| \lesssim_{a,d,N} e^{c_{d,a}y^2}\left[|w|^{-|\operatorname{Re}(\nu)|} \wedge |w|^{-1/2}(1\wedge(\operatorname{Re}(w))^{-N})\right] \quad \operatorname{Re} w>0,\ N\in\mathbb{N}$$

$$\lesssim_{a,d} e^{c_{d,a}y^2}|x-y|^{-1/2} \quad |w|=|x-y|,\ |\operatorname{Re}\nu|\le 1/2$$

$\Rightarrow$ plugging this in $|\hat{m}_\lambda(x)| \lesssim_{a,d} \frac{2^{1-a}}{|\Gamma(a-iy)|}\left(\frac{|z|}{|x|^2}\right)^{\frac{d/2-a}{2}} e^{c_{d,a}y^2}|x|^{-1/2}$

$$\lesssim_{d,a} e^{c_{d,a}y^2}|x|^{a-(d+1)/2}|z|^{\frac{d}{4}-\frac{a}{2}}$$ which is bdd in $|x|$ if $a=\frac{d+1}{2}$

$\square$

Thm 4.22 (Frank-Sabin - extension to Schatten ideals)

Let $d \geq 2$ and assume $q \in \begin{cases} [4/3, 3/2] & d=2 \\ [\frac{d}{2}, \frac{d+1}{2}] & d \geq 3 \end{cases}$ $\qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{p'}$ in above analysis

$\Rightarrow$ For all $z \in \mathbb{C} \setminus [0,\infty)$, we have

$$\|W_1 (-\Delta - z)^{-1} W_2\|_{\mathfrak{S}^{(d-1)q/(d-q)}(L^2(\mathbb{R}^d))} \lesssim |z|^{-1+\frac{d}{2q}} \|W_1\|_{L^{2q}} \|W_2\|_{L^{2q}}$$

and for $\gamma \geq \frac{1}{2}$, $\delta(z) := \operatorname{dist}(z, [0,\infty))$, and all $z \in \mathbb{C} \setminus [0,\infty)$, we have

$$\underset{\substack{\uparrow \\ \text{useful for Lieb-Thirring (complex)}}}{\|W_1 (-\Delta - z)^{-1} W_2\|_{\mathfrak{S}^{2(\gamma+d/2)}(L^2(\mathbb{R}^d))}} \lesssim \delta(z)^{-1+\frac{(d+1)/2}{2(\gamma+d/2)}} |z|^{-\frac{1}{2(\gamma+d/2)}} \|W_1\|_{L^{2(\gamma+d/2)}} \|W_2\|_{L^{2(\gamma+d/2)}}$$

Pf Consider $V_1^{a-it} (-\Delta - z)^{-a+it} W_2^{a-it}$, $t \in \mathbb{R}$. If $a=0$, we recover $L^2 \to L^2$-bounds,

i.e., $\|W_1^{-it} (-\Delta - z)^{it} W_2^{-it}\|_{L^2, L^2} \leq \|W_1\|_{L^\infty} \|W_2\|_{L^\infty}$.

On the other hand, we shall prove

$$\|W_1^{a-it} (-\Delta - z)^{-a+it} W_2^{a-it}\|_{\mathfrak{S}^2} \leq M_{d,a}\, e^{c_{d,a} t^2} \|W_1\|^a_{L^{4ad/(d-1+2a)}} \|W_2\|^a_{L^{4ad/(d-1+2a)}}, \quad t \in \mathbb{R},$$

$$a \in \begin{cases} [1, \frac{3}{2}] & d=2 \\ [\underset{\substack{\uparrow \\ d/2\,?}}{\frac{d-1}{2}}, \frac{d+1}{2}] & d \geq 3 \end{cases}$$

$\Rightarrow$ by complex interpolation in Schatten ideals, one obtains

$\|W_1 (-\Delta - z)^{-1} W_2\|_{\mathfrak{S}^{2a}} \lesssim \|W_1\|_{L^{4ad/(d-1+2a)}} \|W_2\|_{L^{4ad/(d-1+2a)}}$, which is the claimed estimate if one sets $a = q(d-1)/(2(d-q))$. (In fact, if $d=2,3$, then the Hilbert-Schmidt bound is actually already the desired bound and complex interpolation is unnecessary)

The $\mathfrak{S}^2$-bound for $a = \frac{d+1}{2}$ follows from the previous computation where the kernel $(-\Delta - z)^{-a+it}(x,y)$ is uniformly bdd, in which case one merely integrates

$$\int dx dy\, |W_1(x)|^{2a} |W_2(y)|^{2a} = \|W_1\|^{d+1}_{d+1} \|W_2\|^{d+1}_{d+1} \qquad \left(\text{observe } \frac{4d \cdot \frac{d+1}{2}}{d-1+d+1} = d+1 \ \checkmark\right)$$

On the other hand if $a \in (\frac{d-1}{2}, \frac{d+1}{2}]$, then $|(-\Delta - z)^{-a+it}(x,y)| \lesssim |x|^{a-\frac{d+1}{2}} \cdot e^{ct^2}$

and so, by Hardy-Littlewood-Sobolev and Hölder,

$$\int dx dy\, |W_1(x)|^{2a} |W_2(y)|^a |x-y|^{2a-d-1} \lesssim \|W_1\|^{2a}_{L^{4ad/(d-1+2a)}} \|W_2\|^{2a}_{L^{4ad/(d-1+2a)}} \text{ if } a \in [\underset{\substack{\uparrow \\ d/2\,?}}{\frac{d-1}{2}}, \frac{d+1}{2}$$

The second bound follow from complex interpolation between the first bound for $q = \frac{d+1}{2}$ ($\Leftrightarrow \gamma = \frac{1}{2}$) and $\|W_1 (-\Delta - z)^{-1} W_2\| \leq \delta(z)^{-1} \|W_1\|_\infty \|W_2\|_\infty$ for $\gamma = \infty$. $\blacksquare$

4.5 Tomas-Stein for quadratic, non-compact surfaces – Strichartz estimates

We shall, for concreteness, only consider the paraboloid $\mathbb{P}^{d+1} = \{(\xi', (\xi')^2), \xi' \in \mathbb{R}^d\}$

$$(F_S^* g)(x) = \int_{\mathbb{P}^{d+1}} \frac{d\sigma_{\mathbb{P}^{d+1}}(\xi)}{2|\xi|} e^{2\pi i x \cdot \xi} g(\xi) = \int_{\mathbb{R}^d} d\xi \; e^{2\pi i (x' \cdot \xi' + x_{d+1} (\xi')^2)} g(\xi', (\xi')^2) \cdot \frac{1}{2((\xi')^2 + (\xi')^4)^{1/2}}$$

~~$\mathbb{P}^{d+1} = \{\xi' \in \mathbb{R}^d, \xi'^2$~~ $\mathbb{P}^{d+1} = \{\xi \in \mathbb{R}^{d+1} ; \xi_{d+1} = (\xi')^2\}$ Write $n = d+1$ and let $d\sigma$ be

$\to S_\rho = \{\xi \in \mathbb{R}^n ; \xi^2 = r\}$

the Lebesgue measure on ~~$\mathbb{P}$~~ S_ρ and $d\mu(\xi) = |\nabla(\xi^4)^2|^{-1} d\sigma(\xi) = (2|\xi|)^{-1} d\sigma(\xi)$. (Leray measure)

Then as in the proof of the endpoint Tomas-Stein theorem, consider a family of tempered distributions $(G_z)_{z \in \mathbb{C}}$ on $\mathbb{R}^n$ s.t.

$$(G_z, \varphi) = g(z) \int_{\mathbb{R}^n} (\xi^2 - r)_+^z \varphi(\xi) d\xi \; := \; g(z) \int_{\mathbb{R}} d\rho \; (\rho - r)_+^z \int_{S_\rho} d\mu_\rho(\xi) \; \varphi(\xi)$$

$$d\xi = d\rho \, d\mu_\rho(\xi), \quad d\mu_\rho(\xi) = \frac{d\sigma_\rho(\xi)}{|\nabla P(\xi)|} \quad \text{Leray measure}$$

tr

The function $g(z)$ is chosen similarly as in the proof of Tomas-Stein; crucially though, it must have a simple zero at $z = -1$ to ensure $G_{-1} \equiv \delta_{S_r} = d\mu_r$

The estimate $\|F_S f\|_{L^2(\mathbb{P}^n)} \lesssim \|f\|_{L^{\frac{2(n+1)}{n+3}}}$ then again follows from complex interpolation between $T_{iy} : L^2 \to L^2$ and $T_{-\lambda_0} : L^1 \to L^\infty$ where T_z denotes convolution with integral kernel given by the FT of G_z.

In particular, by the ~~mach~~ Prop 4.11 (Frank-Sabin) one obtains an upgrade to the Schatten bound $\|W_1 F_S^* F_S W_2\|_{\gamma^{n+1}(L^2(\mathbb{R}^n))} \lesssim \|W_1\|_{L^{n+1}} \|W_2\|_{L^{n+1}}$

A well known consequence of this restriction estimate ~~as~~ is the following dispersive estimate for the Schrödinger evolution

Thm 4.23 (Strichartz estimate)

Let $u(x,t)$ be a solution of the inhomogeneous free Schrödinger evolution in d spatial dimensions

$$\begin{cases} i\partial_t u(x,t) = -\Delta_x u(x,t) + g(x,t) \\ u(x,0) = f(x), \quad t = 0 \end{cases}$$

Suppose $f \in L^2(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^{d+1})$ with $p = \frac{2(n+1)}{n+3}$ and $n = d+1$. Then $u \in L^q(\mathbb{R}^{d+1})$ for $q = \frac{2(n+1)}{n-1}$ with $n = d+1$ and $\|u\|_{L^q(\mathbb{R}^{d+1})} \lesssim \|f\|_2 + \|g\|_p$

~~Pf (HW)~~

Corollary. By a reformulation of Prop 4.11 which is in fact equivalent to

$$\Big\| \sum_{j\in J} \nu_j |Af_j|^2 \Big\|_{L^{p'/2}(\mathbb{R}^d)} \lesssim \Big(\sum_{j\in J} |\nu_j|^{\alpha'} \Big)^{1/\alpha'}, \quad (\nu_j)_{j\in J} \subseteq \mathbb{C}, \ \cancel{(f_j)}$$

$$\Leftrightarrow \| W A A^* W \|_{\mathcal{G}^{\alpha}(L^2(\mathbb{R}^d))} \lesssim \|W\|^2_{L^{2p/(2-p)}(\mathbb{R}^d)} \qquad \begin{array}{ll} A : L^2 \to L^{p'} & (A = \mathcal{F}_S^*) \\ A^* : L^p \to L^2 & (A^* = \mathcal{F}_S) \end{array}$$

and the Schatten upgrade of Strichartz, one actually gets dispersive estimates for collections of orthonormal functions

Let $\frac{2}{p} + \frac{d}{q} = d$, $1 \leq q < 1 + \frac{2}{d-1}$ and $(f_j)_{j\in J} \subseteq L^2(\mathbb{R}^d)$ an (possibly infinite) system of orthonormal functions ($\sim$ fermions) and $(\nu_j)_{j\in J} \subseteq \mathbb{C}$ complex coefficients, then $\Big\| \sum_{j\in J} \nu_j |e^{it\Delta} f_j|^2 \Big\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \Big(\sum_j |\nu_j|^{\frac{2q}{q+1}} \Big)^{\frac{q+1}{2q}}$

$\Leftrightarrow \| \rho_{e^{it\Delta}\gamma e^{-it\Delta}} \|_{L^p_t L^q_x} \lesssim \|\gamma\|_{\mathcal{G}^{2q/(q+1)}(L^2(\mathbb{R}^d))}$ whenever $\gamma = \sum_{j\in J} \nu_j |f_j\rangle\langle f_j|$ where $\rho_{e^{it\Delta}\gamma e^{-it\Delta}}$ is the density of $e^{it\Delta}\gamma e^{-it\Delta}$ defined by duality as

$$\operatorname{tr}(\gamma e^{-it\Delta} K e^{it\Delta}) \equiv \int \rho_{e^{it\Delta}\gamma e^{-it\Delta}}(x) K(x) dx$$

$\forall \gamma \in \mathcal{G}^{\alpha'}$ $(\alpha' = \frac{2q}{q+1})$ and ~~$K \in \mathcal{G}^{\alpha}$~~, $K = W^2$, $W \in L^{p/(2-p)}$ above

Pf of Thm 4.23 (HW)

By Duhamel's principle
$$u(x,t) = \int_0^t e^{i(t-s)\Delta} g(\cdot,s)\,ds + \underbrace{\int_{\mathbb{R}^d} \hat{f}(\xi) e^{-2\pi i x\cdot\xi + it|\xi|^2} d\xi}_{\cancel{(\mathcal{F}_S^* f)(x)}\ (\mathcal{F}_S^* f)(x,t) \in L^q_{x,t}(\mathbb{R}^n)}$$

$q = \frac{2(n+1)}{n-1}$

whenever $f \in L^2(\mathbb{R}^{n-1})$

To handle the first term, recall $\|e^{it\Delta}h\|_2 = \|h\|_2$, $\|e^{it\Delta}h\|_\infty \lesssim t^{-d/2}\|h\|_1$ and so $\|e^{it\Delta}h\|_q \lesssim t^{-r}\|h\|_{q'}$

where $\frac{1}{q'} - \frac{1}{q} = \theta$

$r = \theta d/2 = \frac{d}{2}\left(\frac{2}{q'} - 1\right)$

$= d\left(\frac{1}{q'} - \frac{1}{2}\right) = \frac{d}{d+2}$

$$\Rightarrow \Big\| \int_0^t e^{i(t-s)\Delta} g(\cdot,s)\,ds \Big\|_{L^q_x} \leq \int_0^t ds\, \|e^{i(t-s)\Delta} g(\cdot,s)\|_{L^q_x}$$

$$\leq \int_0^t ds\, |t-s|^{-d/(d+2)} \|g(\cdot,s)\|_{L^{q'}_x} \overset{HLS}{\leq} \|g(\cdot,s)\|_{L^{q'}_{x,t}(\mathbb{R}^{d+1})}$$

$1 + \frac{d-d-4}{2(d+2)} = \frac{2d}{2(d+2)} = \frac{d}{d+2}$ ✓

$\frac{d+2}{d} \cdot 2 \quad \frac{2(d+2)}{d+4} \Leftrightarrow \boxed{\frac{2(d+2)}{d}}$

Direct application of Strichartz estimates (Thm 4.23) to well-posedness of NLS

(Many textbooks on NLS available → see, e.g., Tao, Cazenave, Sogge, Hörmander, Linares-Ponce, ...)

here $\begin{cases} i\partial_t u + \Delta u = \lambda|u|^2 u & \text{in } ~~d~~ \mathbb{R}^{2+1} \\ u(0) = f & \lambda \in \mathbb{C} \end{cases}$ — mass-critical NLS; consider ~~$u_\mu(x,t)$~~ =

$\uparrow$ generally $|u|^{4/d}$

$u_\mu(x,t) = \mu^{-\frac{d}{2}} u(\frac{x}{\mu}, \frac{t}{\mu^2})$ then $\|u_\mu(\cdot,t)\|_2 = \|u(\cdot,\frac{t}{\mu^2})\|_2$

$$\partial_t u_\mu = \overbrace{\mu^{-d/2-2}}^{\mu^{-3}} (\partial_t u)(\tfrac{x}{\mu}, \tfrac{t}{\mu^2})$$

$$\Delta u_\mu = \underbrace{\mu^{-d/2-2}}_{\mu^{-3}} (\Delta u)(\tfrac{x}{\mu}, \tfrac{t}{\mu^2})$$

$$\lambda|u|^2 u = \lambda \cdot \underbrace{\mu^{-3d/2}}_{\mu^{-3}} u(\tfrac{x}{\mu}, \tfrac{t}{\mu^2})$$

$\Rightarrow \lambda$ cannot be scaled away!

To get an idea why Schrödinger's equation is dispersive, let us pretend that Δu was absent. Then we can integrate the equation and obtain

$$u(t) = (|u(0)|^{-2} - 2at)^{-(a+ib)/(2a)} \frac{u(0)}{|u(0)|}$$

where $-i\lambda = a+ib$. The solution will obviously blow up in finite time if $a = \operatorname{Im}\lambda > 0$, namely at $t = (2a|u(0)|^2)^{-1}$. Basically, the non-linearity creates a positive feedback loop and leads to a rapid increase of u. However $-\Delta$ actually pushes "large chunks" of u to $|x| \to \infty$.

Thm 4.24 Suppose $\|f\|_{L^2} = 1$. If λ is sufficiently small, then there exists a global solution u to 2d-NLS s.t. $\|u\|_2 \lesssim 1$ $\forall t$ and the dispersive / space-time estimate $\|u\|_{L^4_{x,t}} \lesssim 1$ holds. ($\frac{2(d+2)}{d} = 4$) This solution is unique subject to the above condition, and the solution depends continuously on the norms mentioned on the initial datum $f \in L^2$. Finally, the solution scatters, i.e. $\exists f_+ \in L^2$ (initial dat.) s.t. $\|u(t) - e^{-it\Delta} f_+\|_2 \xrightarrow{t\to\infty} 0$. ("2d - L^2-crit NLS is globally well posed")

Note that the sign of λ is irrelevant, i.e., it doesn't matter whether the equation is repulsive (usually the easier case) or attractive (which has a potential to let solutions blow up in finite time)

Connection with restriction is obvious: if $b = 0$ then global solution given by

$$u(x,t) = \widehat{f d\sigma}(x,t) = \int d\xi\, e^{-2\pi i(x\cdot\xi + t\xi^2)} f(\xi)$$

since $i\partial_t \tilde{u}(\xi,t) = +\xi^2 \tilde{u}(\xi,t) \overset{\text{FT in } t}{\Leftrightarrow} +\tau \hat{u}(\xi,\tau) = +\xi^2 \hat{u}(\xi,\tau) \Leftrightarrow \tau = \xi^2$

~~$\Rightarrow \tilde{u}(\xi,t) = f\, e^{it\xi^2} f(\xi)$~~ $\Rightarrow u(x,t) = e^{+it\Delta} f(x) \Rightarrow \tilde{u}(\xi,t) = e^{-it\xi^2} \hat{f}(\xi)$

To prove Thm 4.24 we use again Duhamel to write the solution as

$$u(t) = e^{+it\Delta} f + i\lambda \int_0^t ds\, e^{+i(t-s)\Delta} (|u|^2 u)(s)$$

To find a solution, we shall use an iterative method whose convergence will be ensured by a fixed point theorem, once we've establisched a certain contraction map (Finding this map in application is usually tedious ~ trial and error)

We start with the zeroth approximation which is the linear approximation

$$u_0(t) = e^{+it\Delta} f$$

and then improve this by incorporating the non-linearity by plugging this easy solution into Duhamel's formula, i.e,

$$u_1(t) = e^{+it\Delta} f + i\lambda \int_0^t ds\ e^{+i(t-s)\Delta} (|u_0|^2 u_0)(s)$$

and so forth by defining $u_{n+1} \equiv N u_n$ where

$$((N(u))(t) := e^{+it\Delta} f + i\lambda \int_0^t e^{+i(t-s)\Delta} (|u|^2 u)(s)\, ds$$

This iteration converges, once we show that $N: X \to X$ is a contraction in a suitable Banach space X which contains the solution $u(x,t)$. We shall chose $X = \{u : \|u\|_{L^4_{x,t}} \le C\}$ since we expect u to disperse as time goes by, at least for small couplings of the non-linearity.

So we ~~we~~ want to show $\|u_0\|_{L^4_{x,t}} \lesssim 1$ and $\|N(u) - N(v)\|_{L^4_{x,t}} \lesssim \|u-v\|_{L^4_{x,t}}$ $\forall u,v \in L^4_{x,t}$

$\|u_0\|_{L^4_{x,t}} = \|e^{+it\Delta} f\|_{L^4_{x,t}} \lesssim \|f\|_{L^2}$ by Strichartz' version of Tomas-Stein.

$$|N(u) - N(v)| \le \underset{\substack{\nearrow \\ \text{will have} \\ \text{to be small}}}{i\lambda} \int_0^t ds\ e^{+i(t-s)\Delta} \underbrace{\left[\, |u|^2 u - |v|^2 v \,\right]}_{\underset{HW}{\le} \ldots \le \frac{3}{2} [\, |u|^2 |u-v| + |v|^2 |u-v| \,]}$$

Assume for the moment that we knew $\| \int^t ds\ e^{+i(t-s)\Delta} F(s) \|_{L^4_{x,t}} \lesssim \|F\|_{L^{4/3}_{x,t}}$ ("retarded Strichartz")

then $\|N(u) - N(v)\| \lesssim |\lambda| \left(\| |u|^2 |u-v| \|_{L^{4/3}_{x,t}} + \| |v|^2 |u-v| \|_{L^{4/3}_{x,t}} \right) \lesssim |\lambda|\, \|u\|^2_{L^4_{x,t}} \|u-v\|_{L^4_{x,t}}$ as desired.

By the fixed point theorem, we've proven global existence, uniqueness, and continuous dependence on the initial data. As a bonus, we obtain $\|u\|_{L^4_{x,t}} \lesssim 1$

We're left to show the retarded Strichartz estimate, $u \in L^2_x$, and that u scatters.

$u \in L^2_x$ follows from Duhamel, i.e, $\|u(t)\|_{L^2_x} \le \|e^{it\Delta} f\|_{L^2_x} + \| \int_0^t ds\ e^{+i(t-s)\Delta} |u|^2 u \|_{L^2}$

Besides the retarded Strichartz, there's the dual homogeneous Strichartz estimate which says $\| \int_0^{\infty} ds\ e^{-is\Delta} F(s) \|_{L^2} \lesssim \|F\|_{L^{4/3}_{x,t}}$. This shows that the second summand is bdd by $\| |u|^2 u \|_{L^{4/3}_{x,t}} = \|u\|^3_{L^4_{x,t}}$. (by pulling out $\|e^{it\Delta}\|_{22} = 1$)

We shall now show scattering. Define $f_+ = f + i\lambda \int_0^\infty e^{-is\Delta}(|u|^2u)(s)$, i.e., f_+ equals f but backdates the effect of the non-linearity. By the dual homogeneous Strichartz estimate ~~and the argument~~ we again see $f_+ \in L^2$. We wish to show $\| u(t) - e^{+it\Delta} f_+ \|_{L^2} \to 0$ as $t \to \infty$. We again use Duhamel to write

$$u(t) - e^{+it\Delta} f_+ = e^{+it\Delta} f + i\lambda \int_0^t ds\ e^{+i(t-s)\Delta}(|u|^2 u)(s) - \left(e^{+it\Delta} f + i\lambda \int_0^\infty ds\ e^{+i(t-s)\Delta} (|u|^2 u)(s) \right)$$

$$= -i\lambda e^{+it\Delta} \int_0^\infty ds\ e^{-is\Delta} (|u|^2 u)(s)\, \mathbb{1}_{[t,\infty)}(s)$$

$$\Rightarrow \| u(t) - e^{+it\Delta} f_+ \|_{L^2_x} \underset{\substack{\text{dual homog.}\\ \text{Strichartz}}}{\lesssim} |\lambda|\, \| \mathbb{1}_{[t,\infty)} |u|^2 u \|_{L^{4/3}_{x,t}}$$

which gives the claim upon passing to $t \to \infty$ and using monotone convergence.

$\Rightarrow$ Left to prove dual homog. Strichartz $\| \int_0^\infty ds\ e^{-is\Delta} F(s) \|_{L^2_x} \lesssim \| F \|_{L^{4/3}_{x,t}}$ and

retarded Strichartz $\| \int_0^t ds\ e^{+i(t-s)\Delta} F(s) \|_{L^4_{x,t}} \lesssim \| F \|_{L^{4/3}_{x,t}}$.

First, the dual homog. Strichartz follows from retarded Strichartz (but implies actually the classic Strichartz ineq $\Rightarrow$ HW).

Squaring the dual homog. Strichartz gives

$$\left\langle \int_0^\infty dt\ e^{-it\Delta} F(t), \int_0^\infty ds\ e^{-is\Delta} F(s) \right\rangle = \int_0^\infty dt \int_0^\infty ds\ \langle F(t), e^{-i(s-t)\Delta} F(s) \rangle$$

By symmetry, it suffices to consider the portion $s \in [0, t]$. Writing out the scalar product, we consider

$$\left| \int_{\mathbb{R}^2} dx \int_0^\infty dt\ \overline{F(t,x)} \int_0^t ds \left(e^{-i(s-t)\Delta} F(s) \right)(x) \right| \underset{\substack{\text{Hölder in}\\ (x,t)}}{\leq} \| F \|_{L^{4/3}_{x,t}} \underbrace{\left\| \int_0^t ds\ e^{-i(s-t)\Delta} F(s) \right\|_{L^4_{x,t}}}_{\lesssim \| F \|_{L^{4/3}_{x,t}} \text{ by retarded Strichartz}}$$

Thus, we're left with retarded Strichartz.
Writing out the $L^4_{x,t}$ norm gives

$$\left(\int_0^\infty dt\ \left\| \int_0^t ds\ e^{i(t-s)\Delta} F(s) \right\|^4_{L^4_x} \right)^{1/4} \overset{\text{Minkowski}}{\leq} \left(\int_0^\infty dt \left(\int_0^t ds\ \underbrace{\| e^{i(t-s)\Delta} F(s) \|_{L^4_x}}_{\lesssim |t-s|^{-1/2} \text{ by interpolating between } \| e^{it\Delta} \|_{2,2} = 1 \text{ and } \| e^{it\Delta} \|_{1\to\infty} \lesssim |t|^{-d/2} = |t|^{-1} \text{ in } d=2} \right)^4 \right)^{1/4}$$

$$\lesssim \left(\int_0^\infty dt \left(\int_0^t ds\ |t-s|^{-1/2} \| F(s) \|_{L^{4/3}_x} \right)^4 \right)^{1/4}$$

$$\underset{\substack{\text{(replace } t \text{ in upper}\\ \text{integration limit}\\ \text{by } \infty}}{\leq} \left\| |\cdot|^{-1/2} * \| F(\cdot) \|_{L^{4/3}_x} \right\|_{L^4_t} \underset{\text{HLS}}{\lesssim} \| F \|_{L^{4/3}_{x,t}}$$

▨

4.6 (Complex) eigenvalue bounds and Lieb-Thirring

Heuristically, the ~~sum~~ number of negative eigenvalues of a Schrödinger operator $T(D) - \lambda V(x)$ in $L^2(\mathbb{R}^d)$, say, should be given by the semiclassical phase space integral, when the kinetic energy becomes very weak compared to the potential, i.e., $\lambda \to \infty$.

$N_e(\lambda V) := \#\{$eigenvalues of $T(D) - \lambda V$ below $-e\}$ for a given $e > 0$

$$N_e(\lambda V) \sim \int_{T(p)-\lambda V(q) < -e} dp\, dq \; 1$$; if $e = 0$, and $T(D) = -\Delta$, then

$$\int dq \int_{p^2 < \lambda V(q)} dp \sim \int dq \, (\lambda V(q))^{d/2}$$

and similarly the sum of neg ev $\sim \int dq \int dp \, (T(p) - \lambda V(q))_-$ ($A_- = \max\{0,-A\}$ ≥ 0)

$$\underset{T(p)=p^2}{\sim} \int dq \, (\lambda V(q))^{1+d/2}$$

The $\sim$ relations can be made precise and for $\lambda \to \infty$ these relations become exact up to errors of order $o(\lambda^{d/2})$ or $o(\lambda^{d/2+1})$, respectively. These asymptotics go back to celebrated works of Weyl, Avakumović, Hörmander, and many others.
Obtaining sharp non-asymptotic results for $\mathrm{tr}(T(D) - \lambda V)_-^\gamma$ for fixed λ and $\gamma \geq$ are highly non-trivial, sharp meaning, getting the best constant for such bounds.
Bounds of the form $\mathrm{tr}(T(D) - \overset{\lambda=1}{\lambda} V)_-^\gamma \leq F(\gamma, \|V\|_{L^{p(\gamma)}})$ go back to works by Lieb and Thirring in the context of proving stellar and quantum mechanical stability of matter.
Here we shall not be concerned with the best constants but rather review one classic standard argument that leads to Lieb-Thirring bounds.

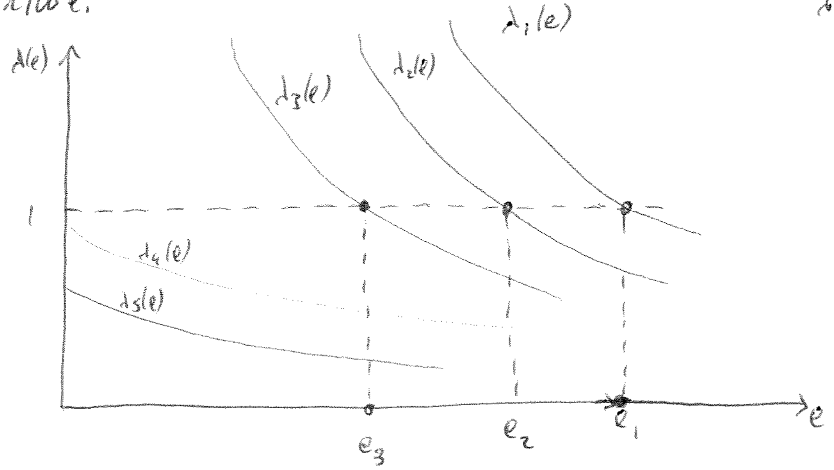
4.6.1 Birman-Schwinger principle $(V > 0)$

$$(T(D) - V)\psi = -e\psi \Leftrightarrow (T(D)+e)V^{-\frac{1}{2}}\underbrace{V^{\frac{1}{2}}\psi}_{\varphi} = V^{\frac{1}{2}}\underbrace{V^{\frac{1}{2}}\psi}_{\varphi} \Leftrightarrow \underbrace{V^{\frac{1}{2}}(T(D)+e)^{-1}V^{\frac{1}{2}}}_{= BS(e)}\varphi = \varphi$$

$\rightarrow$ $T(D) - V$ has an ev $-e < 0 \Leftrightarrow BS(e)$ has an ev $= 1$.

Let $\lambda(e)$ denote the eigenvalues of $BS(e)$. For $T(\xi) \geq 0$, non-decreasing, it's easy to see that $\|BS(e)\|$ and the $\lambda(e)$ decrease as e increases. We get the following picture.

$\lambda_1(e) \geq \lambda_2(e) \geq \dots$



$\Rightarrow$ # of eigenvalues of $T(D) - V$ below $-e$

$=$ # of eigenvalues of $BS(e)$ above 1,

$\Rightarrow$ Summing (powers) of the eigenvalues of $BS(e)$ above 1 gives

$$N_e(V) \leq \operatorname{tr} BS(e)^m \quad \text{for all } m > 0$$

which in turn can be used to get bounds on moments of eigenvalues by

~~$|e_j|^\gamma = \gamma \int_0^{|e_j|}$~~ $\quad (e_j)^\gamma = \gamma \int_0^{e_j} e^{\gamma-1} de \rightarrow \sum_j (e_j)^\gamma = \gamma \int_0^\infty de \; e^{\gamma-1} \underbrace{\sum_j \Theta(e_j - e)}$

bad at ∞ or 0

$= $ # of ev's $-e_j$ below $-e$

$= N_e(V)$

$\cancel{\leq} = N_{e/2}(V - \frac{e}{2})$

$\overset{(*)}{\leq} N_{e/2}(\max\{0, V - \frac{e}{2}\})$

$= N_{e/2}((V - \frac{e}{2})_+)$

Lieb/Araki

$$\operatorname{tr} BS(e)^m \overset{!}{\leq} \operatorname{tr} V^{\frac{m}{2}}(T(D)+e)^{-m}V^{\frac{m}{2}}$$

$$\left(= \int dx dy \; V(x)^{m/2} V(y)^{m/2} (T(D)+e)^{-m}(x-y)\right)$$

$$\overset{FT}{=} \int dx \; V(x)^m \int d\xi \; (T(\xi)+e)^{-m}$$

For $T(D) = -\Delta$, $\operatorname{tr} BS(e)^m \leq \int dx \; V(x)^m \int d\xi \; (\xi^2 + e)^{-m} \underset{m > d/2}{\sim} \int dx \; V(x)^m e^{-m+d/2}$

$\uparrow$ together with $e^{\gamma-1}$ not integrable

$$\rightarrow N_e(V) \overset{(*)}{\leq} \int de \; e^{\gamma-1-m+d/2} \int dx \; (V(x) - \tfrac{e}{2})_+^m = c \int dx \; V(x)^{\gamma+d/2}$$

$\uparrow$ will cut off e-Integral for large e

if $\gamma - m + \frac{d}{2} > 0$

~~$\Leftrightarrow \gamma > \frac{d}{2} - m$ (e.g. $m = \frac{d}{2}$)~~

$\gamma + \frac{d}{2} > m > \frac{d}{2} \rightarrow \underline{\underline{\gamma > 0}}$

($\gamma = 0 \rightarrow$ Cwikel; not here)

Evidently the BS principle can be used to get ev bands also for non-self-adjoint ops.

Recall KRS-bound $\| V^{\frac12}(-\Delta - z)^{-1} |V|^{\frac12} \|_{L^2,L^2} \overset{(*)}{\lesssim} |z|^{-1+\frac{d}{2}(\overbrace{\frac1p - \frac1{p'}}^{=1/q})} \|V\|_{L^q}$

If $\|BS(z)\| > 1$, then $-\Delta - V$ must have $-z$ as an eigenvalue and hence, by it satisfies $|z|^{1-d/(2q)} \lesssim \|V\|_{L^q}$ whenever $q \in [\underset{\underset{\gamma=0}{\uparrow}}{\frac{d}{2}}, \underset{\underset{\gamma=\frac12}{\uparrow}}{\frac{d+1}{2}}]$

Higher q are known when V satisfies additional symmetry restrictions, such as spherical symmetries. However, Bögli showed that for such bounds to hold in general, $q \le d$, i.e., $\gamma \le d/2$ is necessary. It is open to decide whether $\gamma \in (\frac12, \frac{d}{2}]$ are actually admissible; Laptev and Safronov conjectured this to be true and for spherically symmetric potentials it is indeed the case, see Frank-Simon if $\gamma < d/2$

Anyway, using the extension of KRS to Schatten ideals (Thm 4.22) one can say something about eigenvalue sums outside certain sectors, but we shall not pursue this here any further.

Instead, we shall go back to the self-adjoint case and see how the TS theorem in Schatten ideals may give LT estimates for more complicated ~~operat~~ kinetic energies such as $|\Delta + 1|$.

$$\| V^{\frac12}(T+e)^{-1} V^{\frac12}\|_m^m \le \| V^{\frac12}(T+e)^{-1}\chi(T) V^{\frac12}\|_m^m + \underbrace{\| V^{\frac12}(T+e)^{-1}_{\overset{\wedge}{(1-\chi(T))}} V^{\frac12}\|_m^m}$$

$$\overset{\text{Seiler-Simon}}{\le} \|V\|_m^m \int_{T(\xi)>1} \frac{d\xi}{(\xi^2+e)^m} \sim \|V\|_m^m e^{\frac{d}{2}-m}$$

↳ trace inequality would lead us to

$$\int_0^1 dt \frac{\| V^{m/2} F_{s_t}^* F_{s_t} V^{m/2}\|_1}{(t+e)^m} \sim (e^{1-m} \wedge e^{-m}) \|V\|_m^m \quad \text{making use of } \underbrace{\| F_{s_t}^* V^{m/2} \|_{\gamma^2}}_{(\int d\xi \int dx\, |V|^m)^{1/2}} = \|V\|_m^{\frac{m}{2}}$$

on the other hand, using Cor 4.12 we obtain

$$\| V^{\frac12}(T+e)^{-1}\chi(T) V^{\frac12}\|_{\gamma_{(d-1)q/(d-q)}} \le \int_0^1 \frac{dt}{t+e} \underbrace{\| V^{\frac12} F_{s_t}^* F_{s_t} V^{\frac12}\|_{\gamma_{q(d-1)/(d-q)}}}_{\lesssim \|V\|_{L^q}}$$

$$\lesssim (e^{-1} \wedge \log 1/e) \|V\|_q .$$