I Uncertainty principle / Preliminaries

$\mathcal{F}_1$ (FT on L^1)

Reminder on some specifics of FT on $\mathbb{R}^d$

Let $f \in L^1(\mathbb{R}^d)$, then define $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)\,dx \equiv (\mathcal{F}_1 f)(\xi)$

This notion can be generalized to treat finite, complex-valued measures μ on $\mathbb{R}^d$ by setting $\hat{\mu}(\xi) = \int e^{-2\pi i x\cdot\xi}\,d\mu(x)$

Recall that the set of such measures, denoted by $M(\mathbb{R}^d)$ is equipped with the norm $\|\mu\| = |\mu|(\mathbb{R}^d)$ where $|\mu|$ is the total variation ($|\mu|(E) := \sup_{\pi} \sum_{A\in\pi} |\mu(A)|$ $\forall E \in \Sigma$

π: partitions of E into a countable number of disjoint measurable subsets; Σ: Borel-σ-Alg.

Recall total variation is a positive measure

In particular, $L^1(\mathbb{R}^d) \subseteq M(\mathbb{R}^d)$ via the identification

$$f \to \mu, \quad d\mu = f\,dx$$

<u>Examples</u> • Let δ_a be the Dirac measure located at $a \in \mathbb{R}^d$,

i.e., $\delta_a(E) = \begin{cases} 0 & E \not\ni a \\ 1 & E \ni a \end{cases}$. Then $\widehat{\delta_a}(\xi) = e^{-2\pi i a\cdot\xi}$

• $\mathcal{F}(e^{-\pi x^2})(\xi) = e^{-\pi\xi^2}$ (HW)

<u>Proposition 1.1</u> If $\mu \in M(\mathbb{R}^d)$, then $\underbrace{\|\hat{\mu}\|_\infty}_{L^\infty(\mathbb{R}^d)} \le \|\mu\|_{M(\mathbb{R}^d)} = |\mu|(\mathbb{R}^d)$ and $\hat{\mu}$ is continuous.

<u>Pf</u> • $|\hat{\mu}(\xi)| \le \int d|\mu|(x) = |\mu|(\mathbb{R}^d) = \|\mu\|_{M(\mathbb{R}^d)}$

• $\hat{\mu}(\xi+h) = \int \underbrace{e^{-2\pi i x(\xi+h)}}_{\xrightarrow[h\to 0]{\text{ptwise}} e^{-2\pi i x\cdot\xi}} d\mu(x) \longrightarrow \hat{\mu}(\xi)$ by dominated convergence.

<u>Basic properties</u> ($\tau \in \mathbb{R}^d$)

• $f_\tau(x) := f(x-\tau) \rightsquigarrow \hat{f}_\tau(\xi) = e^{-2\pi i\tau\cdot\xi}\hat{f}(\xi)$

• $e_\tau(x) := e^{2\pi i x\cdot\tau} \rightsquigarrow \widehat{e_\tau f}(\xi) = \hat{f}(\xi-\tau)$

• Let $T: \mathbb{R}^d \to \mathbb{R}^d$ be a linear, invertible map.

$$\rightsquigarrow (f\circ T)^\wedge(\xi) = \int e^{-2\pi i x\cdot\xi} f(Tx)\,dx = \frac{1}{|\det T|}\int e^{-2\pi i \langle T^{-1}x,\xi\rangle} f(x)\,dx$$

$$= |\det T|^{-1}\hat{f}(T^{-t}\xi)$$

If $T \in O(d)$ (orthogonal), then $TT^t = \mathbb{1}$, $|\det T| = 1$ and hence $(f\circ T)^\wedge = \hat{f}\circ T$

I.p. if f radial, then $\hat{f}$ radial since orthogonal transformations act transitively on spheres.

If T is a dilation, i.e., $Tx = rx$, $r>0$, then $(f\circ T)^\wedge = r^{-d}\hat{f}\circ T^{-1}$

i.e. $f(r\cdot)^\wedge(\xi) = r^{-d}\hat{f}(\xi/r)$.

Classic uncertainty principle (there are numerous such principles, even whole books have been dedicated to them; most notably perhaps Wolff, Folland-Sitaram, Fefferman, Havin-Jöricke)

<u>Proposition 1.2</u> Let $R>0$ and $f\in C_c^\infty(\mathbb{R}^d)$ be such that $\operatorname{supp} f \subseteq \bar{B}_0(R) \equiv \{x\in\mathbb{R}^d : |x|\le R\}$. Then $\hat{f}$ is holomorphic and satisfies $|\hat{f}(\xi)| \le e^{2\pi R|\operatorname{Im}\xi|}\|f\|_{L^1}$ for $\xi\in\mathbb{C}$

Moreover, $\operatorname{supp}\hat{f}$ cannot be compact, unless $\hat{f}\equiv 0$.

<u>Pf</u> HW

<u>~~Remar~~ ~~Theor~~ Prop 1.3</u> Suppose $\mu\in M(\mathbb{R}^d)$, $\operatorname{supp}\mu$ compact. Then $\hat{\mu}\in C^\infty$ and $D^\alpha\hat{\mu} = ((-2\pi i x)^\alpha\mu)^\wedge$ (any $\alpha\in\mathbb{N}_0^d$)

Furthermore, if $\operatorname{supp}\mu\subseteq B_0(R)$, then $\|D^\alpha\hat{\mu}\|_\infty \le (2\pi R)^{|\alpha|}\|\mu\|$

<u>Pf</u>
- $\|D^\alpha\hat{\mu}\|_\infty \le (2\pi R)^{|\alpha|}\|\mu\|$ obvious from $D^\alpha\hat{\mu} = ((-2\pi i x)^\alpha\mu)^\wedge$ and Prop 1.2
- first consider $|\alpha|=1$ and show $\hat{\mu}\in C^1$. By induction, the ~~hy~~ statement follows since also $x^\alpha\mu$ satisfies ~~assumptions of~~ hypotheses.
- $$\frac{\hat{\mu}(\xi+he_j)-\hat{\mu}(\xi)}{h} = \int \underbrace{\frac{e^{-2\pi i x\cdot he_j}-1}{h}}_{\xrightarrow{h\to 0}\, -2\pi i x_j \text{ pointwise and } |\ldots|\lesssim |x_j| \text{ which is } \mu\text{-integrable}} e^{-2\pi i x\cdot\xi}\,d\mu(x) \longrightarrow (-2\pi i x_j\mu)^\wedge(\xi) \text{ by dominated convergence}$$

☒

~~by dominated~~

localization in physical space $\Rightarrow$ smoothness in frequency space!

But the converse heuristic is true as well

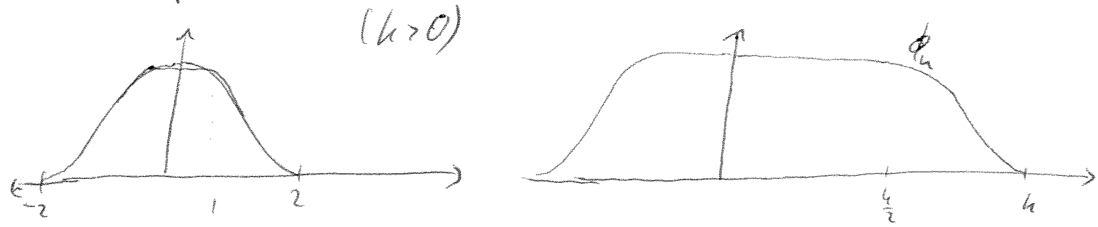
<u>Prop 1.4</u> Suppose $f\in C^N$ and $D^\alpha f\in L^1$ $\forall\, 0\le|\alpha|\le N$ $\Rightarrow (D^\alpha f)^\wedge(\xi) = (2\pi i\xi)^\alpha\hat{f}(\xi)$, $|\alpha|\le N$

and $|\hat{f}(\xi)| \lesssim (1+|\xi|)^{-N}$

The proof uses integration by parts, which is most easily justified if f is compactly supported

Let $\phi \in C_c^\infty$ be a radial bump fct centered at 0 $\left(\begin{matrix} \phi(x)=1 & |x|<1 \\ \phi(x)=0 & |x|>2 \end{matrix} \quad 0 \le \phi \le 1 \right)$ ③

and define $\phi_h(x) = \phi(x/h)$ $(h>0)$ ~~i.e.~~ $\cancel{(D^\alpha \phi_h)(x) \lesssim h^{-|\alpha|}}$



In particular $D^\alpha \phi_h = 0$ if $|x| \notin [h, 2h]$ and $|(D^\alpha \phi_h)(x)| \lesssim h^{-|\alpha|}$

<u>Lemma 1.5</u> If $f \in C^N$, $D^\alpha f \in L^1$ $\forall$ $0 \le |\alpha| \le N$ and if we let $f_h := f \phi_h$, then

$$\|D^\alpha f_h - D^\alpha f\|_{L^1} \xrightarrow{h\to\infty} 0 \quad \forall \ |\alpha| \le N$$

<u>Pf</u> (<u>HW</u>) $D^\alpha f_h = D^\alpha(\phi_h f) = \phi_h D^\alpha f + \sum_{0<\beta\le\alpha} \underbrace{c_\beta}_{\text{some constants}} (D^{\alpha-\beta} f)\, D^\beta \phi_h$

- $\|\phi_h D^\alpha f - D^\alpha f\|_{L^1} \xrightarrow{h\to\infty} 0$ obvious, so suffices to check $\underbrace{\|(D^{\alpha-\beta} f)\, D^\beta \phi_h\|_{L^1}}_{\le \underbrace{\|D^{\alpha-\beta} f\|_{L^1(|x|>h)}}_{\xrightarrow{h\to\infty} 0} \underbrace{\|D^\beta \phi_h\|_{L^\infty}}_{\lesssim h^{-1} \text{ at least} \xrightarrow{h\to\infty} 0}} \to 0$

$\boxed{}$

<u>Pf of Prop 1.4</u> First assume $f \in C_c^N$ (i.e. with compact support)

- ~~If $f \in C_c^1$~~. Then $\int \frac{\partial f}{\partial x_j}(x) e^{-2\pi i x\cdot\xi} dx = 2\pi i \xi_j \int e^{-2\pi i x\cdot\xi} f(x) dx$, i.e.,

  $(D^\alpha f)^\wedge = (2\pi i \xi)^\alpha \hat{f}(\xi)$ by induction.

- Since $\xi^\alpha \hat{f} \in L^\infty$ (by Prop 1.1 and $D^\alpha f \in L^1$) and $(1+|\xi|)^N \lesssim \sum_{0\le|\alpha|\le N} |\xi^\alpha| \lesssim (1+|\xi|)^N$,

  we have $|\hat{f}(\xi)| \lesssim (1+|\xi|)^{-N}$.

- Now we remove the compact support hypothesis. Let $f_h = \phi_h f$, then the assertion holds for f_h i.e., $(D^\alpha f_h)^\wedge(\xi) = (2\pi i \xi)^\alpha \hat{f}_h(\xi)$

  ~~However~~ $\cancel{\|(2\pi i \xi)^\alpha (\hat{f}_h - \hat{f})\|_{L^\infty} \le \|D^\alpha(f_h - f)\|_{L^1} \to 0}$ ~~by Lemma 1.5~~

  However $\|\widehat{D^\alpha f_h} - \widehat{D^\alpha f}\|_{L^\infty} \le \|D^\alpha(f_h - f)\|_{L^1} \to 0$ by Lemma 1.5

  and $\|\hat{f}_h - \hat{f}\|_{L^\infty} \le \|f_h - f\|_{L^1} \to 0 \Rightarrow \underbrace{(2\pi i \xi)^\alpha \hat{f}_h}_{= (D^\alpha f_h)^\wedge} \to (2\pi i \xi)^\alpha \hat{f}$ ptwise

$\boxed{}$

Remarks 1.6 Prop. 1.2 can be refined in several directions: let $f \in L^1(\mathbb{R}^d)$, $d=1$ (4)

(Complex analysis) a) If $|f(x)| \lesssim_{f,M} e^{-M|x|}$ $\forall x \in \mathbb{R}$, $M>0$ (i.e. f decays super-exponentially), then $\hat{f}$ extends uniquely to an entire fct (i.e. holo on whole $\mathbb{C}$)

decay ⇓ smoothness

b) If $|f(x)| \lesssim_{f,a} e^{-2\pi a|x|}$ $\forall x \in \mathbb{R}$ and some $a>0$, then $\hat{f}$ extends uniquely to a holomorphic fct on the strip $\{\xi \in \mathbb{R} : |\mathrm{Im}(\xi)| < a\}$ and obeys the bound $|\hat{f}(\xi)| \lesssim \frac{1}{a - |\mathrm{Im}\,\xi|}$ [Thm 4.3.1 in Stein-Shakarchi

(holomorphicity follows from Morera + Fubini-Tonelli
uniqueness by analytic continuation + Cauchy integral formula)

On the other hand Prop 1.4 can be refined as well.

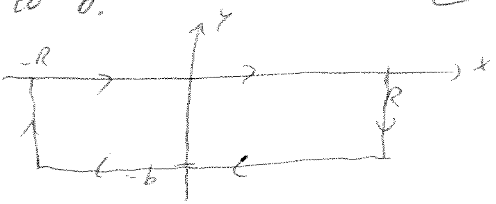
smoothness ⇓ decay

Suppose f is holo on $S_a = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < a\}$ and $|f(x+iy)| \le \frac{1}{1+x^2}$ $\forall x \in \mathbb{R}$, $|y|<a$

(e.g. $f = e^{-\pi z^2}$ or $f(z) = \frac{a}{a^2+z^2}$ (complex Poisson kernel)) (actually $\langle x \rangle^{-1-\epsilon}$ suffices)

$\Rightarrow |\hat{f}(\xi)| \lesssim e^{-2\pi b|\xi|}$ $\forall\, 0 \le b < a$, $\xi \in \mathbb{R}$

Pf $b=0$ obvious, since $f \in L^1(\mathbb{R})$
$0<b<a$; suppose $\xi > 0$; main step consists in shifting down the contour of integration, i.e., the real line, down to b.



~~→ consider~~ $\cancel{g(z) = f(z) e^{-2\pi i \xi \cdot z}}$

we shall show $\underline{f(\cdot + w)^\wedge = e^{2\pi i w \cdot}\, \hat{f}}$, $w \in S_a$,

respectively $\int f(x+w) e^{-2\pi i x\cdot\xi} dx = e^{2\pi i w\cdot\xi} \int f(x) e^{-2\pi i x \cdot \xi} dx$ whenever $|\mathrm{Im}\, w| < a$, $\xi \in \mathbb{R}$

Clearly, $\int f(x) e^{-2\pi i x\cdot\xi} dx = \lim_{R\to\infty} \int_{\gamma_{-R\to R}} f(z) e^{-2\pi i z\cdot\xi} dz$. ~~But by Cauchy, we can also take the detour~~ $\cancel{\int_{\gamma_{-R\to R}} = -\int_{\gamma_{R\to R-ib}} - \int_{\gamma_{R-ib\to -R-ib}} - \int_{\gamma_{-R-ib\to -R}}}$

Similarly, $e^{-2\pi i w\xi} \int f(x+w) e^{-2\pi i x\cdot\xi} dx = \lim_{R\to\infty} \underbrace{\int_{\gamma_{-R+w\to R+w}}} f(z) e^{-2\pi i z\cdot\xi} dz$ by changing variables

$= \int_{\gamma_{-R\to R}} + \int_{\gamma_{R\to R+w}} - \int_{\gamma_{-R\to -R+w}}$

→ suffices to show $\underbrace{\int_{\gamma_{R\to R+w}} f(z) e^{-2\pi i z\cdot\xi} dz} \xrightarrow{R\to\infty} 0$

$= w e^{-2\pi i R\xi} \int_0^1 dt\, \underbrace{f(R+tw)}_{\lesssim \langle R \rangle^{-1-\epsilon}} e^{-2\pi i \xi\cdot tw}$ and $\xrightarrow{R\to\infty} 0$ by dominated convergence

$\Rightarrow$ take $w = \pm ib$ ⇝ $e^{\mp 2\pi b\xi} \hat{f}(\xi) = (f(\cdot \pm ib))^\wedge \Rightarrow |\hat{f}(\xi)| \le e^{\pm 2\pi b\xi} \|\hat{f}\|_\infty$ ☐

The Paley-Wiener thm sharpens the heuristic

smoothness in physical space $\leftrightarrow$ localization in frequency space

Stein-Shakarchi, Rudin, Hörmander..

Thm (Paley-Wiener) Let $f:\mathbb{R}\to\mathbb{C}$ be continuous with $|f(x)|\lesssim(1+|x|)^{-2}$ $\forall x\in\mathbb{R}$

Let $M>0$, then the following are equivalent

(i) $\hat{f}$ is supported on $[-M,M]$

(ii) f extends analytically to an <u>entire</u> fct and obeys $|f(z)|\lesssim e^{2\pi M|z|}$

(iii) f extends analytically to an entire fct that obeys $|f(z)|\lesssim e^{2\pi M|\mathrm{Im}\,z|}$

Pf (i) $\Rightarrow$ (iii) clear (see Prop 12)

(iii) $\Rightarrow$ (ii) clear

(ii) $\Rightarrow$ (i) Assume first the stronger bound $|f(z)|\lesssim\frac{e^{2\pi M|\mathrm{Im}\,z|}}{1+|z|^2}$ $(*)$, $z\in\mathbb{C}$.

Then by complex contour shifting as above,

$$\hat{f}(\xi)=e^{-2\pi b\xi}\int_{\mathbb{R}}f(x-ib)e^{-2\pi ix\cdot\xi}dx,\quad \xi,b\in\mathbb{R}$$

we see $|\hat{f}(\xi)|\lesssim e^{-2\pi b\xi+2\pi Mb}$ (by triangle ineq + $|f(z)|\lesssim\frac{e^{2\pi Mb}}{1+|z|^2}$)

$\Rightarrow$ if $\xi\gtrsim M$, take $b\to\pm\infty$ to conclude $\hat{f}=0$ if $|\xi|>M$

Now we remove the mitigating decay factor $(1+|z|^2)^{-1}$ and consider

$f_\epsilon(z)=\frac{f(z)}{(1+i\epsilon z)^2}$ which is still holo for $\mathrm{Im}\,z\leq0$ and obeys a bound for each $\epsilon>0$; $|f_\epsilon(z)|\lesssim(1+(\mathrm{Re}\,z)^2)^{-1}e^{2\pi M|\mathrm{Im}\,z|}$ of the form $(*)$. Repeating the above steps that one can still conclude $\hat{f}_\epsilon(\xi)=0$ for $\xi>M$, despite the lack of holomorphicity in $\mathrm{Im}\,z>0$.

Similarly, replacing $(1+i\epsilon z)^{-2}$ by $(1-i\epsilon z)^{-2}$ shows $\hat{f}_\epsilon(\xi)=0$ for $\xi<-M$

Now we need to let $\epsilon\to0$ which is however fine ~~because of~~ using dominated convergence and the fact $|f_\epsilon(z)|\lesssim_\epsilon\frac{\exp(2\pi M|\mathrm{Im}\,z|)}{1+(\mathrm{Re}\,z)^2}$

$(1+i\epsilon(x-ib))^{-2}$

$=\int(1+\epsilon b-i\epsilon x)^{-2}dx$

$\sim 1+\int_{|x|>\frac{1}{\epsilon}}(\epsilon x)^{-2}dx$

and $|\hat{f}_\epsilon(\xi)-\hat{f}(\xi)|\lesssim\int\underbrace{|f(x)|}_{<(|x|+1)^{-2}}\Big[\underbrace{\frac{1}{(1-i\epsilon x)^2}-1}_{\xrightarrow{\text{ptwise}}0}\Big]\to0$

(let first $b\to\pm\infty$ and then $\epsilon\to0$)

Thm 3.4 in Stein-Shakarchi, prove it for $\arg z\in(-\frac{\pi}{4},\frac{\pi}{4})$

(ii) $\Rightarrow$ (iii) uses Phragmen-Lindelöf in the following form: let F be holo in $\arg z\in(\alpha,\beta)$ and continuous on boundary. Suppose $|F(z)|\leq1$ on boundary a $|F(z)|\leq e^{c|z|^a}$ in interior $a\in(0,\frac{\pi}{\beta-\alpha})$

$\Rightarrow |F(z)|$ in whole sector

$<\beta<\alpha+2\pi$

here: $|f(x)|<\frac{1}{1+x^2}<1$ on real axis and $|f(iy)|<e^{2\pi M|y|}$ $\Rightarrow$ $f(z)<e^{2\pi M|\mathrm{Im}\,z|}$

$<e^{2\pi M|x|}$ (by taking $F(z)=f(z)e^{2\pi iMz}$)

Fourier Inversion + Plancherel

recall convolution $(\phi * f)(x) = \int_{\mathbb{R}^d} \phi(y) f(x-y)\,dy = (f*\phi)(x)$

$$\operatorname{supp} \phi * f \subset \operatorname{supp}\phi + \operatorname{supp} f \equiv \{x+y : x \in \operatorname{supp}\phi,\ y \in \operatorname{supp} f\}$$
(Minkowski sum)

- the above integral is meaningful whenever
  - $f \in L^p$, $\phi \in L^q$ $\rightsquigarrow$ $\|f*\phi\|_r \le \|f\|_p \|\phi\|_q$, $1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$, $1 \le p, q \le r \le \infty$
    if $r=\infty$, then $f*\phi$ continuous
  - $\phi \in C_c$ and $f \in L^1_{loc}$ and $\phi * f$ is continuous

- recall also

Lemma 1.7 If $\phi \in C_c^\infty$ and $f \in L^1_{loc}$, then $\phi * f \in C^\infty$ and $D^\alpha(\phi * f) = (D^\alpha \phi) * f$

i.p. if $\phi, f \in \mathcal{S} \Rightarrow \phi * f \in \mathcal{S}$.

(Many more properties, e.g., in Hörmander I)

- Interaction with Fourier transform: $\widehat{f*g} = \hat{f}\,\hat{g}$, $f,g \in L^1$
  $\widehat{fg} = \hat{f} * \hat{g}$, $f,g \in \mathcal{S}$

Fundamental issue in Fourier analysis: when do we have a Fourier inversion formula? I.e, given "Fourier coefficients $\hat{f}(\xi)$" (coming from $\int f(x) e^{-2\pi i x\cdot\xi}dx$, $f \in L^1$) when does the integral $\int \hat{f}(\xi) e^{2\pi i x\cdot \xi} d\xi$ make sense and converge to f?

Simple answer

Thm 1.8 If $f \in L^1$ and $\hat{f} \in L^1 \Rightarrow f(x) = \int \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$ for almost every x

Among others, the proof uses the following fact which will be used frequently (Hence, we shall prove it)

Lemma 1.9 Let $\mu, \nu \in M(\mathbb{R}^d)$, then $\int \hat{\mu}\, d\nu = \int \hat{\nu}\, d\mu$.

In particular, if $f, g \in L^1$, then $\int \hat{f}(x) g(x) dx = \int f(x) \hat{g}(x) dx$

Pf By Fubini, $\int \hat{\mu}\, d\nu = \iint e^{-2\pi i x\cdot\xi} d\mu(x)\, d\nu(\xi) = \iint e^{-2\pi i x\cdot \xi} d\nu(\xi)\, d\mu(x) = \int \hat{\nu}\, d\mu$. ☐

Corollary 1.10 If $f \in L^1$ with $\hat{f} = 0$, then $f = 0$.

Thm 1.11 $\mathcal{F}_1 : \mathcal{S}(\mathbb{R}^d) \xrightarrow{\text{onto}} \mathcal{S}(\mathbb{R}^d)$ and for $f, g \in \mathcal{S}$, we have Plancherel $\int \overline{f(x)} g(x)\, dx$
$= \int \overline{\hat{f}(\xi)}\, \hat{g}(\xi)\, d\xi$

Pf Let $f \in \mathcal{S}$, then $f \in L^1$, i.e., $\hat{f} \in L^\infty$

Since $D^\alpha x^\beta f \in \mathcal{S}$ as well, so is $D^\alpha x^\beta f \in L^1 \Rightarrow \widehat{D^\alpha x^\beta f} \in L^\infty$ with

$$\widehat{D^\alpha x^\beta f}(\xi) = (2\pi i)^{|\alpha|}(-2\pi i)^{-|\beta|} \xi^\alpha D_\xi^\beta \hat{f}(\xi)$$

and hence $\xi^\alpha D_\xi^\beta \hat{f} \in L^\infty \Rightarrow \hat{f} \in \mathcal{S}$

Now given $f \in \mathcal{S}$ let $F(x) = f(-x)$ and $g = \hat{F} \in \mathcal{S}$. Since $\hat{\hat{f}}(x) = f(-x)$ a.e. by the L^1-Fourier inversion thm, we have $\hat{g}(x) = \hat{\hat{F}}(x) = F(-x) = f(x)$ ☐

Finally Plancherel: $\int \overline{f(x)} g(x) = \int \overline{f(x)}\, \hat{\hat{g}}(-x) = \int \overline{f(-x)}\, \hat{\hat{g}}(x)$, ~~i.e., $\int \overline{f(x)} g(x) = \int \widehat{\overline{f(-x)}}$~~

i.e., by Lemma 1.9, $\int \overline{f(x)} g(x) = \int \hat{g}\, \underbrace{\mathcal{F}(\overline{f(-\cdot)})}_{= \overline{\mathcal{F}_1 f}}$ ☐

→ Thm 1.11 says that the FT, restricted to $\mathcal{S}$-fcts., is an isometry in L^2 norm. Since $\mathcal{S} \subseteq L^2$ dense, this suggests a way of extending the FT to L^2.

Thm 1.12 (Plancherel) There is a unique bounded operator $\mathcal{F} : L^2 \to L^2$ s.t. $\mathcal{F} f = \hat{f}$ when $f \in \mathcal{S}$. Moreover, (i) $\mathcal{F}$ is unitary
(ii) $\mathcal{F} f = \mathcal{F}_1 f$ if $f \in L^1 \cap L^2$

Pf Existence and uniqueness is immediate from Thm 1.11, as is the fact $\|\mathcal{F} f\|_2 = \|f\|_2$. In view of this isometry property, $\operatorname{ran} \mathcal{F}$ is closed. The unitarity of $\mathcal{F}$ follows then, once we show that $\operatorname{ran}\mathcal{F}$ is dense in L^2. (in which case it must be whole L^2)
But density of $\operatorname{ran}\mathcal{F}$ in L^2 is immediate from $\mathcal{F}_1 : \mathcal{S} \to \mathcal{S}$ onto (Thm 1.11) and density of $\mathcal{S}$ in L^2.
It remains to prove $\mathcal{F}|_{L^1 \cap L^2} = \mathcal{F}_1$. For $\mathcal{S}$-fcts this holds by definition, so suppose $f \in L^1 \cap L^2$ which we know can be approximated by a sequence $(g_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}$ in both L^1 and L^2 norm. By Riemann-Lebesgue, $\hat{g}_n \to \hat{f}$ uniformly and by unitarity of $\mathcal{F}_2$, $\hat{g}_n \to \hat{f}$ in L^2 $\Rightarrow$ $\mathcal{F} f = \mathcal{F}_1 f$ ☐

→ All basic formulae, i.e. the transformation formulae, derived for $\mathcal{F}_1$, continue to hold for $\mathcal{F}$.

Advantage of this L^1+L^2 FT is that its domain pretty wide; e.g. $L^p \subset L^1+L^2$ $\forall\, p\in[1,2]$

and $|x|^{-\alpha}\in L^1+L^2$ for all $\frac{d}{2}<\alpha<d$ (tail in L^2, singularity in L^1)

However L^1+L^2 domain is not always sufficient... one could go along Schwarz's or Hörmander's way and define FT for tempered distros. We will follow Wolff's path and instead consider <u>tempered functions</u> $f\in L^1_{loc}$ that satisfy

$$\int (1+|x|)^{-N}|f(x)|\,dx<\infty$$

for some N, i.e. f can have at most polynomial growth at ∞.

Observe that $\phi f\in L^1$ whenever $\phi\in\mathcal{S}$ and f is tempered fct.

Moreover
$$\mathcal{S}\to\mathbb{C},\qquad \phi\mapsto\int\phi f$$
is continuous for all tempered f. Hence $\phi * f$ well-defined and also tempered.

<u>Def</u> If f and g are tempered functions, then <u>g is called distributional FT of f</u>, if $\int g\phi=\int f\hat{\phi}$ for all $\phi\in\mathcal{S}$.

$$(g,\phi)_{\mathcal{S}}=(f,\hat{\phi})_{\mathcal{S}}\ \text{"="}\ (\mathcal{F}_{\mathcal{S}}f,\phi)_{\mathcal{S}}$$

Given f, such a fct g is unique by density properties of $\mathcal{S}$ and ~~previous~~ similar arguments as above. We shall abuse notation and write $g=\hat{f}$.

<u>Prop 1.13</u> Let $0<Re(\alpha)<d$, ~~and~~ and $h_\alpha(x)=\frac{\Gamma(\alpha/2)}{\pi^{\alpha/2}}|x|^{-\alpha}$ with $\Gamma(s)=\int_0^\infty dt\, t^{s-1}e^{-t}$

Then $\widehat{h_\alpha}=h_{d-\alpha}$ in the sense of distributional FT (and L^1+L^2-FT when $\frac{d}{2}<Re\,\alpha<d$)

Pf $h_\alpha(\xi)=\int_0^\infty e^{-\pi t|\xi|^2}t^{\alpha/2}\frac{dt}{t}$. Since $h_\alpha\in L^1_{loc}$ and tempered,

we have
$$\int_{\mathbb{R}^d}d\xi\int_0^\infty\frac{dt}{t}\,t^{\alpha/2}e^{-\pi t|\xi|^2}\int dx\,e^{-2\pi i x\cdot\xi}f(x)\overset{\text{Fubini-Tonelli}}{=}\int dx\,f(x)\int_0^\infty\frac{dt}{t}\,t^{\frac{\alpha}{2}}\underbrace{e^{-\pi|x|^2/t}}_{t^{-d/2}}$$

$$=\int dx\,f(x)\underbrace{\int_0^\infty\frac{dt}{t}\,t^{\frac{d-\alpha}{2}}e^{-t|x|^2\pi}}_{h_{d-\alpha}(x)}\qquad\square$$

Some L^p specifics

Prop 1.14 (Hausdorff–Young) $\|\hat{f}\|_{p'} \le \|f\|_p \quad \forall \; 1 \le p \le 2$

Pf By interpolation

Remark given p, the Lebesgue exponent must be p'.

Let $f_\lambda(x) = f(\lambda x)$, then $\hat{f}_\lambda(\xi) = \lambda^{-d}\hat{f}(\xi/\lambda)$

$\|f_\lambda\|_p = \lambda^{-d/p}\|f\|_p \qquad \|\hat{f}_\lambda\|_q = \lambda^{-d+\frac{d}{q}}\|\hat{f}\|_q \Rightarrow -\frac{d}{p} \stackrel{!}{=} -d + \frac{d}{q} \Leftrightarrow q = p'$

On the other hand, $p \le 2$ is necessary.

To that end let $\phi_j(x) = \phi(x - x_j)$ with $\phi \in C_c^\infty$ and $\{x_j\}_{j=1}^N \subseteq \mathbb{R}^{nd}$ s.t.

$|x_j - x_k| > 2 \operatorname{diam} \operatorname{supp} \phi$

Let $\{\omega_j\}_{j=1}^N \equiv \omega$ be a Rademacher distributed sequence, i.e., ω_j takes values ± 1 with equal ~~prop~~ probability and $\mathbb{E}\,\omega = 0$ and ω_j's are independent of each other (i.e., $\mathbb{E}\,\omega_j \omega_k = \delta_{jk}$)

$$\Rightarrow \Big\|\sum_{j=1}^N \omega_j \phi_j\Big\|_p^p = \int \Big|\sum \omega_j \phi(x - x_j)\Big|^p \underset{\text{disjoint supports of } \phi_j}{=} \sum_{j=1}^N \int |\phi_j|^p = N\|\phi\|_p^p$$

$$\Rightarrow \Big\|\sum \omega_j \phi_j\Big\|_p = \mathbb{E}\,\Big\|\sum \omega_j \phi_j\Big\|_p = N^{\frac{1}{p}}\|\phi\|_p$$

On the other hand, $\cancel{\mathbb{E}_\omega \big\|\sum_j \omega_j \hat{\phi}_j\big\|_{p'}^{p'} \underset{\text{Khinchin}}{\sim} \sum_j \|\hat{\phi}_j\|}$

$$\mathbb{E}_\omega \Big\|\sum_j \omega_j \hat{\phi}_j\Big\|_{p'}^{p'} = \int d\xi \, |\hat{\phi}(\xi)|^{p'} \underbrace{\mathbb{E}_\omega \Big|\sum_{j=1}^N e^{2\pi i \xi \cdot x_j}\omega_j\Big|^{p'}}_{\overset{\text{Khinchin}}{\sim} \left(\sum_{j=1}^N |e^{2\pi i x_j \cdot \xi}|^2\right)^{p'/2} = N^{p'/2}}$$

$\hat{\phi}_j(\xi) = \hat{\phi}(\xi) e^{2\pi i \xi \cdot x_j}$

$$\sim N^{p'/2}\|\hat{\phi}\|_{p'}^{p'}$$

$\Rightarrow$ To have desired inequality, we need to have $\frac{1}{2} \le \frac{1}{p} \Leftrightarrow p \le 2$.

2 Uncertainty principle

Heuristically: if a measure is concentrated on an ellipsoid E, then for many purposes, $\hat{\mu}$ may be regarded as constant on the dual ellipsoid

<u>Prop 2.1</u> (L^2-Bernstein on disc) Let $f \in L^2$ with $\operatorname{supp} \hat{f} \subseteq B_0(R)$. Then $f \in C^\infty$ and

$$\| D^\alpha f \|_2 \le (2\pi R)^{|\alpha|} \| f \|_2$$

<u>Pf</u> $\| D^\alpha f \|_2 = \| \widehat{D^\alpha f} \|_2 = \| (2\pi \xi)^\alpha \hat{f} \| < (2\pi R)^{|\alpha|} \| f \|_2 \quad \forall \alpha \in \mathbb{N}_0^d$ ☐

$\hat{f}$ compactly supported (extremely rapid decay in frequency space) $\Rightarrow$ f as smooth as we please. Similar statements continue to hold in L^p. However, since Plancherel is not available in this case, we need a different argument. (e.g. Mihlin-Hörmander, which would be an overkill here)

<u>Lemma 2.2</u> If $f \in L^1 + L^2$ with $\operatorname{supp} \hat{f} \subseteq B_0(R)$, then there is $\phi \in \mathcal{S}$ such that

$$f = \phi^{R^{-1}} * f \quad \text{where } \phi^{R^{-1}}(x) = R^d \phi(Rx)$$

<u>Pf</u> Indeed, let $\phi \in \mathcal{S}$ s.t. $\hat{\phi} = 1$ for $|\xi| < 1$, then $\widehat{\phi^{R^{-1}}}(\xi) = \hat{\phi}(\xi/R)$ equals 1 for $|\xi| < R$

$$\Rightarrow \hat{f} - \widehat{\phi^{R^{-1}}} \cdot \hat{f} = 0 \Rightarrow f - \phi^{R^{-1}} * f = 0$$ ☐

~~Prop 2.3~~

<u>Lemma 2.3</u> There are radial bump fcts $\hat{\chi}$ s.t. $\chi \ge 0$ and $\chi > \mathbb{1}_{B_0(1)}$.

<u>Pf</u> (HW) Let g be a radial bump supported on $B_0(1)$, say and take

$$\hat{\chi}(\xi) = A^d B\, (g * g)(A\xi) \text{ for some } A, B > 0$$

$$\cancel{\rightarrow \chi(x) = B \cdot \check{g}(x)^2} \quad \chi(x) = B \cdot \check{g}(x/A)^2$$ ☐

<u>Lemma 2.4</u> There exists a radial $0 < \varphi \in \mathcal{S}$ s.t. $\hat{\varphi} \subset (-1/2, 1/2)^d$ with the property that $\sum_{n \in \mathbb{Z}^d} \varphi(x-n) = 1 \quad \forall x \in \mathbb{R}^d$ (Schlag-Shubin-Wolff)

<u>Pf</u> (HW) We only consider $d=1$, the $d \ge 2$ case being (almost) identical.

In Fourier space the claimed partition of unity reads $\underbrace{\sum_{n \in \mathbb{Z}} \hat{\varphi}(\xi) e^{-2\pi i n \xi}}_{= \sum_{k \in \mathbb{Z}} \hat{\varphi}(k) \delta_k(\xi)} = \delta_0(\xi)$ by Poisson summation $\sum \hat{f}(k) = \sum f(n)$

To ensure equality $\sum \hat{\varphi}(k) \delta_k = \delta_0$, it suffices to assume $\operatorname{supp} \hat{\varphi} \subseteq (-\frac{1}{2}, \frac{1}{2})$, and $\hat{\varphi}(0) = 1$.

We're left to show $\varphi > 0$. To that end take any even $\varphi_0 \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp} \hat{\varphi}_0 \subset (-\frac{1}{4}, \frac{1}{4})$ and $\hat{\varphi}_0(0) = 1$. Since φ_0^2 extends to a entire fct on $\mathbb{C}$, we have $\operatorname{Leb}(\{x : \varphi_0^2 = 0\}) = 0$ (unique continuation)

$\Rightarrow \varphi := \varphi_0^2 * \varphi_0^2 > 0$ everywhere and $\hat{\varphi} = (\hat{\varphi}_0 * \hat{\varphi}_0)^2$ is supported in $(-\frac{1}{2}, \frac{1}{2})$

We check that $\hat{\varphi}(0) = 1$ (or $\hat{\varphi}(0) > 0$; to achieve normalization, we introduce constants)

$$\hat{\varphi}(0) = \left(\int \hat{\varphi}_0(\xi)\, \hat{\varphi}_0(-\xi)\, d\xi\right)^2 = \left(\int \hat{\varphi}_0(\xi)^2\, d\xi\right)^2 > 0$$

φ_0 and $\hat{\varphi}_0$ even $\quad$ $\hat{\varphi}_0$ real-valued since φ_0 even

$$\int_{-\infty}^{\infty} \varphi_0(x)\, e^{ix\xi}$$
$$= \int_0^{\infty} \varphi_0(x)\, e^{ix\xi} + \int_{-\infty}^{0} \varphi_0(x)\, e^{ix\xi}$$
$$= \int_0^{\infty} \varphi_0(x)\, e^{ix\xi} + \varphi_0(x)\, e^{-ix\xi}$$

$\Rightarrow$ we showed $\quad \hat{\varphi}(\xi) \sum e^{-2\pi i n \xi} = \sum \hat{\varphi}(k)\, \delta_k(\xi)$

$\rightarrow \sum \varphi(x-n) = \hat{\varphi}(0) > 0 \Rightarrow$ dividing by $\hat{\varphi}(0)$ gives claim $\square$

<u>Prop 2.5</u> (L^p-Bernstein on disc) Let $f \in L^1 + L^2$ and ~~fr~~ $\operatorname{supp} \hat{f} \subset B_0(R)$.

Then (1) $\forall\, \alpha \in \mathbb{N}_0^d$ and $p \in [1, \infty]$, we have $\|D^\alpha f\|_p \lesssim R^{|\alpha|} \|f\|_p$

(2) $\forall\ 1 \le p \le q \le \infty$ we have $\|f\|_q \lesssim R^{d(\frac{1}{p} - \frac{1}{q})} \|f\|_p$

(upgrade low Lebesgue integrability to high Lebesgue integrability, which corresponds to smoothness, at cost of powers of frequency scale)

(if $R \ll 1$, this cost is in fact a gain!)

~~Pf~~

Pf We again use $f = \underset{R^d \varphi(Rx)}{\underset{\|}{\varphi^{R^{-1}}}} * f$ and first observe $\|\varphi^{R^{-1}}\|_r = \text{const}\, R^{d - d/r} = c\, R^{d/r'}$ for all $1 \le r \le \infty$

Note also $\|\nabla \varphi^{R^{-1}}\|_{L^1} = R \|\varphi\|_{L^1}$. $\Rightarrow \|D^\alpha f\|_p = \|(D^\alpha \varphi^{R^{-1}}) * f\|_p \le \|f\|_p \underbrace{\|D^\alpha \varphi^{R^{-1}}\|_{L^1}}_{= \text{const}\, R^{|\alpha|}}$

The second claim follows analogously, i.e.,

$\|f\|_q = \|\varphi^{R^{-1}} * f\|_q \le \|\varphi^{R^{-1}}\|_r \|f\|_p \lesssim R^{d/r'} \|f\|_p$ with $\underset{1 - \frac{1}{r}}{\underset{\|}{\frac{1}{r'}}} = \frac{1}{p} - \frac{1}{q}$ $\quad \square$

$(\Leftrightarrow)\ \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$

We shall extend these $L^p \to L^q$ estimates to more general geometries.

Def An ellipsoid $E \subseteq \mathbb{R}^d$ is a set of the form

$$E = \{ x \in \mathbb{R}^d : \sum_j \frac{|(x-a)\cdot e_j|^2}{r_j^2} \leq 1 \}, \quad \begin{array}{l} a \in \mathbb{R}^d \text{ (center of } E) \\ \{e_j\}_{j=1}^d \text{ some ONB in } \mathbb{R}^d \text{ (axes)} \\ \{r_j\}_{j=1}^d \subseteq \mathbb{R}_+^d \text{ (axis lengths)} \end{array}$$

We say that two ellipsoids E and E^* are dual to each other, if E^* and E have same axes but reciprocal axis lengths, i.e, $r_j \cdot r_j^* = 1 \quad \forall j = 1..d$. Of course the centers are allowed to differ, i.e, given $E \subseteq \mathbb{R}^d$ as above,

$$E^* = \{ x \in \mathbb{R}^d : \sum_j \frac{|(x_j - b_j) e_j|^2}{(r_j^*)^2} = \sum_j r_j^2 \, |(x-b_j)\cdot e_j|^2 \leq 1 \}$$

<u>Prop 2.6</u> (Bernstein for ellipsoids) Let $f \in L^1 + L^2$ with $\operatorname{supp} \hat{f} \subseteq E$ (ellipsoid)

Then $\|f\|_q \lesssim |E|^{\frac{1}{p}-\frac{1}{q}} \|f\|_p$ whenever $1 \leq p \leq q \leq \infty$

Pf Let h be the center of E and let $T: \mathbb{R}^d \to \mathbb{R}^d$ be ~~the~~ a linear map $B_0(1) \to E - h$

Denote $S = T^{-t}$, i.e., $T = S^{-t}$, and let $f_1(x) = e^{-2\pi i h \cdot x} f(x)$ and $g = f_1 \circ S$

$\to \quad \hat{g}(\xi) = |\det S|^{-1} \hat{f}_1(S^{-t}\xi) = |\det S|^{-1} \hat{f}(S^{-t}(\xi + h)) = |\det T| \, \hat{f}(T(\xi + h))$

$\to \hat{g}$ supported in unit ball $\Rightarrow$ Prop 2.5 applicable, i.e., $\|g\|_q \lesssim \|g\|_p$.

Now we just need to compute these norms

$$\|g\|_q = |\det S|^{-1/q} \|f\|_q = |\det T|^{1/q} \|f\|_q = |E|^{1/q} \|f\|_q$$

(distortion = volume)

so $\|g\|_q \lesssim \|g\|_p$ implies $\|f\|_q \lesssim |E|^{\frac{1}{p}-\frac{1}{q}} \|f\|_p$ as desired $\square$

Often we shall also need pointwise statements, like given $\operatorname{supp} \hat{f} \subseteq E$, then f is roughly constant on E^* and decaying away from E^*.

Classic example later: tubes $T = \{\xi \in \mathbb{R}^d : |\xi_1| < R^{-1}, |\xi_j^a| < R^{-1/2}\ j = 2..d\}$

$\to$ dual tube $T^* = \{x \in \mathbb{R}^d : |x_1 - a_1| < R, |x_j - a_j| < R^{1/2}\ j = 2..d\}$

Let $\phi(x) = (1+|x|^2)^{-N/2}$ and $\phi_{E^*}(x) = \phi(T(x-h^*))$ where h^* is center of E^* and $T : E^* - h^* \to B_0(1)$

We could write this also more explicitly as $\phi_{E^*}(x) = \left(1 + \sum_i \frac{|(x-h^*)e_i|^2}{(r_j^*)^2}\right)^{-N}$

Prop 2.7 Let $f \in L^1 + L^2$, $\operatorname{supp} \hat{f} \subseteq E$ (an ellipsoid). Then for any dual ellipsoid E^* and any $z \in E^*$, we have $|f(z)| \leq C_N \frac{1}{|E^*|} \int |f(x)| \phi_{E^*}(x) dx$

Rem If we wanted to know expanded $\hat{f}(\xi) = \sum_{\substack{\vec{n} \in \mathbb{Z}^d \\ |\vec{n}| < R^{1/2}}} f(\xi) \underbrace{\Upsilon(\xi R^{1/2} - \vec{n})}_{\text{supported in a certain translate } \omega,\text{ centered at } c_\omega \text{ of cube } [-\frac{4}{\sqrt{R}}, \frac{4}{\sqrt{R}}]^d}$ where $\sum_{n \in \mathbb{Z}^d} \Upsilon(\xi - n) = 1$, $\operatorname{supp} \Upsilon \subseteq [-4,4]^d$

Suppose $\operatorname{supp} f \subset [-1,1]^d$

later in restriction theory.

Let $\Omega_R = \{\omega\}$ collection of all these cubes which overlap at most $O(1)$ times

Now on each fixed $\omega \in \Omega_R$, expand $f|_\omega$ in Fourier series, ie,

$f(\xi) = \sum_{q \in Q_R} \left(\frac{R^{\frac{1}{2}}}{8}\right)^d \langle f 1_\omega, \exp(-2\pi i \langle c_q, \cdot\rangle)\rangle \exp(2\pi i\, c_q \cdot \xi)$ $\xi \in \omega$, where q ranges over a tiling Q_R of $\mathbb{R}^d$ consisting of cubes of side length $\sqrt{\frac{R}{8}}$

$$\Rightarrow f(\xi) = \sum_{q \in Q_R} \sum_{\omega \in \Omega_R} 8^{-d} \exp(-2\pi i\, c_q \cdot c_\omega) \langle f 1_\omega, e^{-2\pi i \langle c_q, \cdot\rangle}\rangle \boxed{\Upsilon_{q,\omega}(\xi)} \quad \xi \in [-1,1]^d$$

with $\Upsilon_{q,\omega}(\xi) := R^{d/2} \Upsilon(R^{1/2}(\xi - c_\omega)) \exp(2\pi i\, c_q \cdot (c_\omega - \xi))$

$(E \Upsilon_{q,\omega})(x) = \int_{[-1,1]^d} \Upsilon_{q,\omega}(\xi) e^{2\pi i x \cdot \xi} d\sigma(\xi)$

We call $(E \Upsilon_{q,\omega})(x)$ a wave packet located at c_q traveling in direction of the spatial tube $T_{q,\omega} = \{x \in \mathbb{R}^{d+1} : |x' - c_q + 2c_\omega x_{d+1}| < R^{1/2}, |x_{d+1}| < R\}$

Pf of Prop 2.7

Suppose first $E = B_0(1) = E^*$, then $f = \psi * f$, $\psi \in \mathcal{S}$ some Schwartz fct

$\Rightarrow |f(z)| \le \int |\psi(x-z)||f(x)|\,dx \lesssim_N \int |f(x)| (1+|x-z|)^{-N} dx$

$|x-z| > |x| - |z| > |x|/2$ if $|x| > 1/2$

$\lesssim \int |f(x)| (1+|x|)^{-N} dx$

Now suppose wlog E is some ellipsoid, centered at zero and E^* any dual ellipsoid.

Let h^* and T as above, $T: E^* - h^* \to B_0(1)$ and consider ($\det T = |E^*|^{-1}$)

$g(x) = f(\underbrace{T^{-1}x + h^*}_{\in E^* \text{ when } x \in B_0(1)})$ defined on $B_0(1)$

$\to \hat{g}(\xi) = |\det T| \, e^{2\pi i h^* \cdot \xi} \hat{f}(\underbrace{T^t \xi}_{\in E})$

$T^t \xi \in E$

$\Leftrightarrow \xi \in T^{-t} E$

~~Since $\hat{g}$ is supported on E^* and T^{-1}~~

$\Rightarrow$ $\hat{g}$ will be supported on $T^{-t}E$, ie, the unit ball (since $T: E^* - h^* \to B_0(1)$)

$\Rightarrow$ can use previous result telling us $|g(y)| \le \int \phi(x) |g(x)|\,dx$ whenever $y \in B_0(1)$

$\Rightarrow$ going back shows $|f(T^{-1}z + h^*)| \le \int |f(T^{-1}x + h^*)| \, \phi(x)\,dx$

$= \int \phi_{E^*}(x) |f(x)|\,dx \cdot \underbrace{|\det T|}_{= \frac{1}{|E^*|}}$ ☒

<u>Remarks</u> Prop 2.7 is sharp in the sense that the statement

$|f(z)| \le \frac{1}{|2E^*|} \int |f(x)| \phi_{2E^*}(x)\,dx$ when $z \in E^*$ is wrong!

(see also Guth's lectures)

<u>Example</u> $E = E^* = [0,1]$; let $g \in \mathcal{S}$ with $g(0) \neq 0$, supp $\hat{g} \subset [0,1]$,

and consider $f_N(x) = g(x) \left(1 - \frac{x^2}{4}\right)^N$

$\to$ $\hat{f_N}$ are linear combinations of $\hat{g}$ and its derivatives, hence

supp $\hat{f_N}$ = supp $\hat{g}$

· moreover $\hat{f_N}(\xi) \xrightarrow{N\to\infty} 0$ (boundedly) for $\xi \in [-2,2] \setminus \{0\}$

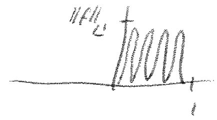
$(1 - x^2/4)^N \sim$ sine or cosine $\Rightarrow \mathcal{FT}$ thereof will focus extremely at zero!)

$\Rightarrow$ supp $\hat{f_N}$ shrinks dramatically

$\Rightarrow$ $|E|$ shrinks and $|E^*|$ grows, but $\int |f(x)| \phi_{E^*} \sim$ const

$\frac{1}{|2E^*|} \int f_N(x) \phi_{E^*}(x)\,dx$ cannot bound $f_N(z)$ for $z \in E^*$ from above

Staying with the example on $[0,1]$, we see that when $\hat f$ is supported on $[0,1]$, then f cannot look like several narrow peaks with width $\sim 10^{-6}$, say,



b.c. $\|f\|_{L^\infty([0,1])} \le \|\hat f\|_{L^1(w_{[0,1]})}$

$\uparrow$ ξ-weight focussed on $[0,1]$

$\rightarrow$ height of peaks $\le \|f\|_{L^\infty}$ however several very narrow peaks cannot add up to the same L^1-norm

$\rightarrow$ f really needs to be $\sim$constant on $[0,1]$

The following prop says that sharp spatial cut-offs lead to frequency smearings on the inverse scale

<u>Prop 2.8</u> ($\rightarrow$ Bourgain 2002, prior to Lemma 3.7 – 3.8)

Let $N_1, N_2 > 0$, $N > N_1 + N_2$, $F : \mathbb{R}^d \to \mathbb{R}^d$ measurable. Let $\gamma_{1/N}(\xi) := N^d \gamma(N\xi)$ where $\check\gamma$ is a smooth bump fct on $\mathbb{R}^d$ with $\check\gamma(x) = 1$ if $|x| < 1$ (i.e., $\gamma_{\frac{1}{N}}$ smoothens on scale $\frac{1}{N}$)

Then $\mathbb{1}_{|x| \le N_1} F(D) \mathbb{1}_{|x|<N_2} = \mathbb{1}_{|x|<N_1} \mathcal{F}^{-1}(F(\xi) * \gamma_{1/N}) \mathcal{F} \mathbb{1}_{|x|<N_2}$

Pf let $f \in \mathcal{S}(\mathbb{R}^d)$, then $\mathbb{1}_{|x|<N_1} F(D) \mathbb{1}_{|x|<N_2} f = \mathbb{1}_{|x|<N_1} \int dy\, \check F(x-y) \mathbb{1}_{|y|<N_2} f(y) = (*)$

Since $|x-y| < N_1 + N_2 < N$, we may smuggle the bump $\check\gamma$ in the above integral, i.e.,

HW

$$(*) = \mathbb{1}_{|x|<N_1} \int dy\, \check F(x-y)\, \check\gamma\left(\tfrac{x-y}{N}\right) \mathbb{1}_{|y|<N_2} f(y) = \mathbb{1}_{|x|<N_1} \mathcal{F}^{-1}(F * \gamma_{\frac{1}{N}}) \mathcal{F} \mathbb{1}_{|x|<N_2} f . \boxtimes$$

<u>Remark</u> We finally mention a more geometric variant of the uncertainty principle by Shubin-Vakilian-Wolff.

We say that a set $E \subseteq \mathbb{R}^d$ is ϵ-thin whenever

$$|E \cap B_x(\rho(x))| \le \epsilon \underbrace{|B_x(\rho(x))|}_{\sim |x|^{-d} \text{ for } |x| \gg 1} \quad \text{where } \rho(x) = \min\{1, \tfrac{1}{|x|}\}$$

<u>Thm (SVW 98)</u> There are $\epsilon > 0$ and $C < \infty$ such that if E and F are two ϵ-thin sets, then for every $f \in L^2(\mathbb{R}^d)$ it holds that

$$\|f\|_{L^2(E^c)} + \|\hat f\|_{L^2(F^c)} \ge C \|f\|_{L^2}$$

$\rightarrow$ f and $\hat f$ cannot both be concentrated on small sets at the same time

This thm is sharp in the sense that the weight fct $\rho(x) = 1 \wedge \frac{1}{|x|}$ cannot be replaced by another one which decays more slowly at ∞.

<u>Example</u> Let $d=1$ and take $\phi \in C_c^\infty$ and let ~~Let~~ $\Phi_N(x) := \sum_{j=-N}^{N} \phi(N(x-j))$

many extremely narrow peaks

will focus on $x=j$ as $N\to\infty$

Then $\hat{\Phi}_N(\xi) = \hat{\phi}\left(\frac{\xi}{N}\right) \cdot N^{-1} \underbrace{\sum_{j=-N}^{N} e^{2\pi i j \xi}}_{= D_N(\xi) = \frac{\sin(2\pi(N+\frac{1}{2})\xi)}{\sin(\pi\xi)}}$ (Dirichlet kernel from theory of Fourier series)

Now let $E_N^A := \bigcup_{j=-AN}^{AN} \left(j - \frac{A}{N}, j + \frac{A}{N}\right)$ and let $F_N^A = (E_N^A)^c$ be its complement

many extremely small intervals ($\sim$ support of Φ_N but also of $\hat{\Phi}_N$ (morally) since $|D_N|^2$ is concentrated on integers and $\hat{\phi}$ decays if N large)

Then one observes that for <u>any</u> $\eta > 0$ there is $A < \infty$ s.t. for any large N, one has $\|\Phi_N\|_{L^2(F_N^A)} + \|\hat{\Phi}_N\|_{L^2(F_N^A)} < \eta \|\Phi_N\|_2$ (contradicts claim)

Now suppose the weight fct ρ is positive, continuous, and satisfies $|x|\rho(x) \xrightarrow{|x|\to\infty} \infty$.

Then for any ϵ and A we have $|B_x(\rho(x)) \cap E_N^A| < \epsilon |B_x(\rho(x))| \; \forall x$ provided N is sufficiently large. (satisfies hypothesis)

$\rightarrow$ the rate fct $\rho(x)$ is indeed optimal

<u>Remark</u> Slowlier decaying ρ maybe admissible if "$\exists \epsilon > 0 \; \exists \, 0 < C < \infty$" is replaced by ~~$\forall \epsilon > 0 \; \exists$~~ "$\forall \epsilon \in (0,1) \; \exists \, C < \infty$".