Fourier Restriction and Applications
Homework Sheet 5

Exercise 5.1

Let σ_1, σ_2 be two Borel measures supported on $M_1, M_2 \subseteq \mathbb{R}^d$. Recall that the convolution is defined by

$$(\sigma_1 * \sigma_2)(\Omega) = \int_{M_1} d\sigma_1(\eta_1) \int_{M_2} d\sigma_2(\eta_2)\, \mathbf{1}_\Omega(\eta_1 + \eta_2)\,, \quad \Omega \subseteq \mathbb{R}^d \text{ measurable}$$

and supported on the Minkowski sum $M_1 + M_2 = \{x + y : x \in M_1, y \in M_2\}$ (cf. Hörmander I). If $\sigma_1 * \sigma_2$ is absolutely continuous with respect to Lebesgue, one can evaluate $\sigma_1 * \sigma_2$ almost everywhere with

$$(\sigma_1 * \sigma_2)(\xi) = \int_{M_1} d\sigma_1(\eta_1) \int_{M_2} d\sigma_2(\eta_2)\, \delta(\xi - \eta_1 - \eta_2) \tag{1}$$

where $\delta(\cdot)$ denotes the usual d-dimensional Dirac delta.

Let $\Omega \subseteq \mathbb{R}^d$ be open, $P \in C^1(\Omega)$ and, $S = \{\xi \in \Omega : P(\xi) = 0\}$ a C^1 codimension one manifold endowed with euclidean surface measure $d\Sigma_S$. Recall that the Leray surface measure of S is given by $d\sigma_S(\xi) = \delta(P(\xi))\, d\xi = |\nabla P(\xi)|^{-1} d\Sigma_S(\xi)$ (in weak sense, i.e., when integrated against test functions).

Suppose $\sigma := \sigma_{\mathbb{S}^{d-1}} = \delta(|\xi|^2 - 1)\, d\xi = 2^{-1}\delta(|\xi| - 1)\, d\xi = 2^{-1} d\Sigma_{\mathbb{S}^{d-1}}(\xi)$ denotes the Leray measure of the unit sphere $\mathbb{S}^{d-1}$. Use (1) to check

$$(\sigma * \sigma)(\xi) = \int_{\mathbb{S}^{d-1}} \delta(1 - |\xi - \eta|^2) d\sigma(\omega) = |\xi|^{-1} \int_{\mathbb{S}^{d-1}} \delta\left(2\frac{\xi}{|\xi|} \cdot \eta - |\xi|\right) d\sigma(\eta)\,,$$

and use spherical coordinates to show

$$(\sigma * \sigma)(\xi) = \frac{|\mathbb{S}^{d-2}|}{2|\xi|} \left(1 - \frac{|\xi|^2}{4}\right)_+^{\frac{d-3}{2}}.$$

For hints, see the nice survey by Foschi and Oliveira e Silva (arXiv 1701.06895).

Exercise 5.2

Let I_1 and I_2 be two θ-arcs on $\mathbb{S}^1$ whose separation is comparable to θ, say

$$I_1 = \{(\cos\varphi, \sin\varphi) : \varphi \in [0, \theta]\}\,, \quad I_2 = \{(\cos(2\theta + \varphi), \sin(2\theta + \varphi)) : \varphi \in [0, \theta]\}\,.$$

Let $d\sigma_{I_j}(\xi) = d\sigma_{\mathbb{S}^1}\big|_{I_j}$ denote the associated Lebesgue surface measures. Show that $\|d\sigma_{I_1} * d\sigma_{I_2}\|_\infty \lesssim \theta^{-1}$.

Hints: You may fatten these arcs up, say $I_1^\varepsilon = \{r(\cos\varphi, \sin\varphi) : r \in [1 - \varepsilon/2, 1 + \varepsilon/2], \varphi \in [0, \theta]\}$, and consider $\varepsilon^{-1}\mathbf{1}_{I^\varepsilon}$ instead of $d\sigma_I$. (Recall $\varepsilon^{-1}\mathbf{1}_{I^\varepsilon} \to d\sigma_I$ weakly.) Show that then $\|\mathbf{1}_{I_1^\varepsilon} * \mathbf{1}_{I_2^\varepsilon}\|_\infty \lesssim \varepsilon^2\theta^{-1}$ (with implicit constant independent of ε) by showing that any translate of I_1^ε can intersect I_2^ε in a geometric body of measure at most $\varepsilon^2\theta^{-1}$. Demonstrate this first in the case where I_j^ε are rectangles with side lengths ε and θ, whose orientations are θ-separated.

Exercise 5.3

The reason why even Lebesgue exponents are helpful in Fourier analysis is that one can often use so-called almost orthogonality arguments.

Definition 0.1. *Let $(\Omega_j)_{j=1}^n$ be a sequence of sets in $\mathbb{R}^d$. We say that "ξ lies in at most $A = A(\xi) \in \mathbb{N}$ of the Ω_j" whenever the maximal number of Ω_j which contain ξ is given by $A(\xi)$, i.e.,*

$$A(\xi) := \sup\{\textit{number of } \ \Omega_j \ \textit{containing } \xi\}\,.$$

Prove the following

Proposition 0.2 (Reverse L^2 and L^4 square function estimates)**.** *Let $f_1, ..., f_n \in \mathcal{S}(\mathbb{R}^d)$ have Fourier support in sets $\Omega_1, ..., \Omega_n \subseteq \mathbb{R}^d$, respectively. Then we have the following assertions.*

1. *(Almost orthogonality) If the sets $\Omega_1, ..., \Omega_n$ have overlap at most A_2, (i.e., every $\xi \in \mathbb{R}^d$ lies in at most $A_2 \in \mathbb{N}$ of the Ω_j, i.e., $\sup_{\xi\in\mathbb{R}^d} A(\xi) \le A_2$) for some $A_2 > 0$, then*

$$\|\sum_{j=1}^n f_j\|_{L^2(\mathbb{R}^d)} \le A_2^{1/2} \|(\sum_{j=1}^n |f_j|^2)^{1/2}\|_{L^2(\mathbb{R}^d)}\,.$$

2. *(Almost bi-orthogonality) If the (n^2) sum sets $\Omega_i + \Omega_j := \{\xi + \xi' : \ \xi \in \Omega\,, \xi' \in \Omega'\}$ with $i, j \in \{1, ..., n\}$ have overlap at most A_4 for some $A_4 > 0$, then*

$$\|\sum_{j=1}^n f_j\|_{L^4(\mathbb{R}^d)} \le A_4^{1/4} \|(\sum_{j=1}^n |f_j|^2)^{1/2}\|_{L^4(\mathbb{R}^d)}\,.$$

Remark 0.3. (1) Clearly, the above theme can be generalized for L^{2p} with $p \in \mathbb{N}$, if one assumes that the sum sets $\sum_{j=1}^p \Omega_j$ have overlap at most A_{2p}, see, e.g., Gressman–Guo–Pierce–Roos–Yung (arXiv 1906.05877).
(2) By using $f_i\overline{f_j}$ in place of $f_i f_j$, one can also establish a variant of (2) in Proposition 0.2 where the sum set $\Omega_i + \Omega_j$ is replaced by the difference set $\Omega_i - \Omega_j := \{\xi - \xi' : \ \xi \in \Omega\,, \xi' \in \Omega'\}$.