Summer term 2021

Fourier Restriction and Applications
Homework Sheet 4

Exercise 4.1
Frank's and Sabin's restriction theorem in Schatten ideals (Corollary 3.12) is predated by observations in the context of scattering amplitudes by Birman, Koplienko, Krein, Kuroda, and Yafaev among others. Prove the following

Theorem 0.1. *Suppose $|V(x)| \lesssim (1+|x|)^{-\rho}$ for some $\rho > 1$ and let*

$$\begin{aligned}\Gamma(\lambda) : \mathcal{S}(\mathbb{R}^d) &\to \mathcal{S}'(\mathbb{S}^{d-1}) \\ f &\mapsto 2^{-1/2}\lambda^{(d-2)/4}\hat{f}(\lambda^{1/2}\cdot)\end{aligned}$$

be the rescaled restriction operator on $\sqrt{\lambda}\mathbb{S}^{d-1}$ with adjoint $\Gamma(\lambda)^*$ (extension operator) given by

$$(\Gamma(\lambda)^* g)(x) = 2^{-1/2}\lambda^{(d-2)/4}\int_{\mathbb{S}^{d-1}} \mathrm{e}^{2\pi i\sqrt{\lambda}x\cdot\xi} g(\xi)\, d\sigma(\xi)$$

where $d\sigma$ is the Lebesgue measure on $\mathbb{S}^{d-1}$. Then for all $\lambda > 0$ and $p > (d-1)/(\rho-1)$ and $p \geq 1$, one has

$$\|\Gamma(\lambda)V\Gamma(\lambda)^*\|_{\mathcal{S}^p(L^2(\mathbb{S}^{d-1}))} \lesssim \lambda^{-1/2+(d-1)/(2p)}\,.$$

<u>Hints:</u> Use the same strategy as in the proof of Corollary 3.12, i.e., interpolation in Schatten ideals between a bounded operator (when $|V(x)|$ decays like $|x|^{-1-\varepsilon}$) and a trace class operator (when $|V(x)|$ decays like $|x|^{-d-\varepsilon}$). (See also Yafaev's textbook *Mathematical Scattering Theory. Analytic Theory.*)

Exercise 4.2
Prove Theorem 3.23, i.e., suppose $u(x,t)$ solves the inhomogeneous free Schrödinger equation

$$\begin{aligned} i\partial_t u(x,t) + \Delta u(x,t) &= g(x,t)\,, \qquad x \in \mathbb{R}^d,\, t \in \mathbb{R} \\ u(x,0) &= f(x) \end{aligned}$$

with $f \in L^2(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^{d+1})$. Show that $\|u\|_{L^q(\mathbb{R}^{d+1})} \lesssim \|f\|_{L^2(\mathbb{R}^d)} + \|g\|_{L^p(\mathbb{R}^{d+1})}$ for $q = 2(d+2)/d$ and $p = 2(d+2)/(d+4)$.
<u>Hints:</u> Use the Tomas–Stein theorem for quadratic surfaces and recall Duhamel's formula

$$u(x,t) = \int_0^t (\mathrm{e}^{i(t-s)\Delta}g)(x,s)\,ds + c\int_{\mathbb{R}^d} \hat{f}(\xi)\mathrm{e}^{-2\pi i x\cdot\xi - it|\xi|^2}\,d\xi\,,$$

(where $c \in \mathbb{C}$ is some constant) and the estimates $\|\mathrm{e}^{it\Delta}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = 1$ and $\|\mathrm{e}^{it\Delta}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2}$.

Exercise 4.3
Use duality of L^p spaces to show that the homogeneous dual Strichartz estimate

$$\Big\|\int_0^\infty ds\,\mathrm{e}^{-is\Delta}F(s)\Big\|_{L^2_x(\mathbb{R}^2)} \lesssim \|F\|_{L^{4/3}_{x,t}(\mathbb{R}^{2+1})}$$

implies the classic Strichartz estimate $\|e^{it\Delta} f\|_{L^4_{x,t}(\mathbb{R}^{2+1})} \lesssim \|f\|_{L^2_x(\mathbb{R}^2)}$.

Exercise 4.4
Prove that the following statements are equivalent to each other. (See also Section 19.3 and Theorem 19.8 in Mattila's book *Fourier Analysis and Hausdorff Dimension*.)

1. $\|\widehat{g\,d\sigma}\|_{L^q(\mathbb{R}^d)} \lesssim \|g\|_{L^p(S)}$ for $q > 2d/(d-1)$ and $q = (d+1)p'/(d-1)$.
2. $\|\widehat{g\,d\sigma}\|_{L^q(\mathbb{R}^d)} \lesssim \|g\|_{L^\infty(S)}$ for $q > 2d/(d-1)$.
3. $\|\widehat{g\,d\sigma}\|_{L^q(\mathbb{R}^d)} \lesssim \|g\|_{L^q(S)}$ for $q > 2d/(d-1)$.