Fourier Restriction and Applications
Homework Sheet 3

Exercise 3.1
Let $\Omega \subseteq \mathbb{R}^{d-1}$ be an open set and $F, \tilde{F} \in C(\Omega : \mathbb{R})$. Let $\alpha > 1/2$ and

$$\theta = \begin{cases} \alpha - 1/2 & \text{if } \alpha < 3/2 \\ 1 - \varepsilon & \text{for all } \varepsilon \in (0,1) \text{ if } \alpha = 3/2 \\ 1 & \text{if } \alpha > 3/2 \end{cases} .$$

Show that

$$\int_\Omega |\hat{u}(\xi', F(\xi')) - \hat{u}(\xi', \tilde{F}(\xi'))|^2 \, d\xi' \lesssim_{\alpha,\theta} \sup_{\xi' \in \Omega} |F(\xi') - \tilde{F}(\xi')|^{2\theta} \int_{\mathbb{R}^d} \langle x_d \rangle^{2\alpha} |u(x)|^2 \, dx \, .$$

Exercise 3.2
Let $1 \leq p, q \leq \infty$ and $S = \{\xi \in \mathbb{R}^d : \xi_d = 0, \, |\xi| \leq 1\}$ with endowed surface measure $d\xi\big|_S = d\xi'$. Suppose $\|\hat{f}\|_{L^q(S)} \lesssim_{p,q,S} \|f\|_{L^p(\mathbb{R}^d)}$ holds for all $f \in \mathcal{S}(\mathbb{R}^d)$. Show that $p = 1$ necessarily.

Exercise 3.3
Let $S = \mathbb{S}^{d-1}$ and $S_t := \{\xi \in \mathbb{R}^d : |\xi| = \sqrt{1+t}\}$ endowed with Lebesgue surface measures $d\sigma_S$ and $d\sigma_{S_t}$, respectively. Show that the non-endpoint Tomas–Stein theorem, and, in fact Hölder continuity, follows from decay estimates of the Fourier transform of the surface measure.

Lemma 0.1. *Let $\tau \in (0,1)$, $0 < \beta \leq (d-1)/2$, $p_\circ = 2(1+\beta)/(2+\beta) \in (1,2)$, and $1 \leq p < p_\circ$, and denote $1/q := 1/p - 1/p'$ and $\widehat{d\omega} := \widehat{d\omega_{S_t}} - \widehat{d\omega_S}$. If there is $\alpha \in (0, \min\{\beta + 1 - q, q\})$ such that*

$$|\widehat{d\omega}(x)| \leq c_\tau \, |t|^\alpha (1 + |x|)^{\alpha - \beta} \tag{1}$$

holds for some $c_\tau > 0$ and all $|t| \in (0, \tau)$, then

$$\sup_{|t| \in (0,\tau)} \|\mathcal{F}_{S_t}^* \mathcal{F}_{S_t} - \mathcal{F}_S^* \mathcal{F}_S\|_{L^p \to L^{p'}} \lesssim_{\alpha,q,\tau} |t|^{\alpha/q} \, . \tag{2}$$

Exercise 3.4
Use the previous exercise to prove Hölder continuity of the non-endpoint Tomas–Stein theorem for $S_t := \{\xi \in \mathbb{R}^d : |\xi| = \sqrt{1+t}\}$ (with $S_0 = S = \mathbb{S}^{d-1}$ as above).

Theorem 0.2. *Let $0 < \tau < 1$, $1 \leq p < 2(d+1)/(d+3)$, $1/q = 1/p - 1/p'$, i.e., $1 \leq q < (d+1)/2$, and $0 < \alpha < \min\{(d+1)/2 - q, q\}$. Then*

$$\sup_{|t| \in (0,\tau)} \|\mathcal{F}_{S_t}^* \mathcal{F}_{S_t} - \mathcal{F}_S^* \mathcal{F}_S\|_{L^p \to L^{p'}} \lesssim_{\alpha,q,\tau} |t|^{\alpha/q} \, . \tag{3}$$