Fourier Restriction and Applications
Homework Sheet 2

Exercise 2.1
Suppose $\varphi \in C^\infty((a,b):\mathbb{R})$ with $\varphi'(x) \neq 0$ and $\psi \in C^\infty((a,b):\mathbb{C})$ not necessarily vanishing at a and b. Find a condition on φ and ψ such that for all $N \in \mathbb{N}$, one has $|\int_a^b \mathrm{e}^{i\lambda\varphi(x)}\psi(x)| \lesssim_{\varphi,\psi,a,b,N} \lambda^{-N}$.

Exercise 2.2
Find a counterexample that shows that a simple lower bound for $|\varphi'(x)|$ alone does not suffice for van der Corput's lemma (Proposition 2.1.2) to hold.

Exercise 2.3
Prove Corollary 2.1.3 in the notes.

Exercise 2.4

1. Let $\ell \in \mathbb{N}_0$. Show that

$$I_\ell(\lambda) = \int_{\mathbb{R}} \mathrm{e}^{i\lambda x^2} x^\ell \mathrm{e}^{-x^2}\,dx \sim \lambda^{-\frac{\ell+1}{2}} \sum_{j\geq 0} c_j^{(\ell)} \lambda^{-j} \quad \text{as } \lambda \to \infty \tag{1}$$

for certain $\{c_j^{(\ell)}\}_{j\in\mathbb{N}_0} \subseteq \mathbb{C}$, i.e., for all $N, r \in \mathbb{N}_0$ one has

$$\left| \left(\frac{d}{d\lambda}\right)^r \left[I_\ell(\lambda) - \lambda^{-\frac{\ell+1}{2}} \sum_{j=0}^{N} c_j^{(\ell)} \lambda^{-j} \right] \right| \lesssim \lambda^{-r-(\frac{\ell+1}{2}+N+1)} \quad \text{as } \lambda \to \infty\,.$$

2. Let $\eta \in C_c^\infty$, $g \in \mathcal{S}(\mathbb{R})$ with $g(x) = 0$ for $|x|$ sufficiently small, and $\ell, N \in \mathbb{N}_0$. Suppose we knew the estimates

$$\left| \int_{\mathbb{R}} \mathrm{e}^{i\lambda x^2} x^\ell \eta(x)\,dx \right| \lesssim_{\eta,\ell} \lambda^{-\frac{\ell+1}{2}} \tag{2}$$

and

$$\left| \int_{\mathbb{R}} \mathrm{e}^{i\lambda x^2} g(x)\,dx \right| \lesssim_{g,N} \lambda^{-N}\,. \tag{3}$$

Suppose $\psi \in C_c^\infty(\mathbb{R})$ is supported in a sufficiently small neighborhood around $x = 0$. Combine (1), (2), and (3) to show

$$\int_{\mathbb{R}} \mathrm{e}^{i\lambda x^2} \psi(x)\,dx \sim \lambda^{-1/2} \sum_{j\geq 0} a_j \lambda^{-j/2} \quad \text{as } \lambda \to \infty$$

for certain $\{a_j\}_{j\in\mathbb{N}_0} \subseteq \mathbb{C}$.

Hint: Multiply and divide the integrand on the left side by e^{-x^2}, write $\psi(x) = \psi(x)\tilde{\psi}(x)$ for some $\tilde{\psi} \in C_c^\infty(\mathbb{R})$ with $\tilde{\psi}(x) = 1$ for $x \in \mathrm{supp}(\psi)$, and expand $\mathrm{e}^{x^2}\psi(x)$ in a Taylor series.