Fourier Restriction and Applications
Homework Sheet 1

Exercise 1.1
Let $R > 0$ and $f \in C_c^\infty(\mathbb{R})$ be such that $\operatorname{supp} f \subseteq \overline{B}_0(R) \equiv \{x \in \mathbb{R} : |x| \leq R\}$. Show that $\hat{f}$ is holomorphic and satisfies $|\hat{f}(\xi)| \leq \mathrm{e}^{2\pi R |\operatorname{Im} \xi|} \|f\|_1$ for $\xi \in \mathbb{C}$. Conclude that $\operatorname{supp} \mathcal{F}[f]$ cannot be compact, unless $\hat{f} \equiv 0$.

Exercise 1.2
Let $\varphi \in C_c^\infty(\mathbb{R}^d : [0,1])$ be a radial bump centered at $x = 0$ with $\varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq 2$, and let $\varphi_k(x) = \varphi(x/k)$. Let $f \in C^N(\mathbb{R}^d)$ with $D^\alpha f \in L^1(\mathbb{R}^d)$ for all $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$ with $0 \leq |\alpha| := \sum_{j=1}^d \alpha_j \leq N$. Show that $\lim_{k \to \infty} \|D^\alpha f_k - D^\alpha f\|_{L^1} = 0$ for all $|\alpha| \leq N$.

Exercise 1.3
Convince yourself that there are radial bump functions $\hat{\chi}$ in Fourier space such that $\chi(x) \geq 0$ and $\chi(x) > \mathbf{1}_{B_0(1)}(x)$ in physical space.

Exercise 1.4
Let $N_1, N_2 > 0$, $N > N_1 + N_2$, and $F : \mathbb{R}^d \to \mathbb{R}^d$ be measurable, bounded. Let $F(D)$ be the associated Fourier multiplier acting as $F(D)\psi(x) = \mathcal{F}^{-1}[F\hat{\psi}](x)$ for $\psi \in \mathcal{S}(\mathbb{R}^d)$, say. Let $\gamma \in C_c^\infty(\mathbb{R}^d : [0,1])$ be a radial smooth bump function in physical space and $\hat{\gamma}_{1/N}(\xi) := N^d \hat{\gamma}(N\xi)$. Show that $\mathbf{1}_{|x| \leq N_1} F(D) \mathbf{1}_{|x| \leq N_2} = \mathbf{1}_{|x| \leq N_1} \mathcal{F}^{-1}(F(\xi) * \gamma_{1/N}) \mathcal{F} \mathbf{1}_{|x| \leq N_2}$.

Remark: The above identity says that (even rough) spatial cut-offs lead to frequency smearing on the inverse scale.

Exercise 1.5
Let T be an invertible $d \times d$ symmetric matrix with complex entries and $\operatorname{Re}(T) \geq 0$. Show that the distributional Fourier transform of $\mathrm{e}^{-\pi \langle x, Tx \rangle}$ is given by $(\det T^{-1})^{1/2} \mathrm{e}^{-\pi \langle x, T^{-1} x \rangle}$.

Remarks: (1) Since the set H of symmetric matrices A with $\operatorname{Re}(A) \geq 0$ is convex, it follows that there is a unique analytic branch of $H \ni A \mapsto (\det A)^{1/2}$ such that $(\det A)^{1/2} > 0$ when A is real.
(2) If $T = iB$ with B being purely real (and invertible), one can compute $(\det T)^{1/2}$ directly. In this case, one can assume that B is diagonal since a real orthogonal transformation does not change $(\det T)^{1/2}$ (since it does not if T is real), i.e., we assume $\langle x, Bx \rangle = \sum_j b_j x_j^2$ with $b_j \in \mathbb{R} \setminus \{0\}$ being the eigenvalues of B. Then $(\det(iB))^{1/2} = |\det B|^{1/2} \mathrm{e}^{\pi i \operatorname{sgn}(B)/4}$ where $\operatorname{sgn}(B) = \sum_j \operatorname{sgn}(b_j)$ denotes the signature of B.

Hints: The computation is a generalization of the scalar case $d = 1$. If you have not yet seen this, see Proposition 4.2 in Wolff's lecture notes or Theorem 7.6.1 in Hörmander's *The Analysis of Linear Partial Differential Operators*.