Fourier restriction and applications (Summer term 2021)

March 22, 2021

Lectures: Wed 16.45-18.15 and Thu 13.15-14.45, starting on April 14, 2021
Exercises: Wed 16.45-18.15 (every second week)

Let S be a smooth hypersurface in $\mathbb{R}^d$ such as the sphere or the (truncated) paraboloid, or the cone with associated surface measure $d\sigma$. The classic trace lemma asserts that whenever $f \in L^2_\sigma(\mathbb{R}^d)$ with $\sigma > 1/2$, then $\hat{f}$ can be meaningfully restricted to S with $\|\hat{f}\|_{L^2(S,d\sigma)} \lesssim \|f\|_{L^2_\sigma(\mathbb{R}^d)}$. The decay of a function at infinity can also be measured in scales of L^p spaces; morally, $f \in L^p$ decays faster the smaller p is. Thus, the question arises whether estimates such as $\|\hat{f}\|_{L^q(S,d\sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$ hold for certain $1 \leq p, q \leq \infty$. If $p = 2$, then $\hat{f} \in L^2$, i.e., $\hat{f}$ belongs to an equivalence class within which its members are allowed to differ off of sets of measure zero. Hence, restriction of L^2 functions to codimension one surfaces is impossible. In the 60's E. Stein made the surprising discovery that when S is *curved*, then one can indeed restrict the Fourier transform of L^p functions (with $p \in [1, 2)$). This finding lead to the *restriction problem*: "For which sets $S \subseteq \mathbb{R}^d$ and which $1 \leq p \leq q \leq \infty$ is there an estimate

$$\|\hat{f}|_S\|_{L^q(S)} \lesssim_{p,q,d} \|f\|_{L^p(\mathbb{R}^d)}$$

for smooth, compactly supported f?"

In its dual form the restriction problem asks for finding p', q' such that the *extension estimate*

$$\|(gd\sigma)^\vee\|_{L^{p'}(\mathbb{R}^d)} \lesssim_{p,q,d} \|g\|_{L^{q'}(S)}, \quad (gd\sigma)^\vee(x) = \int_S \mathrm{e}^{2\pi i x\cdot\xi} g(\xi)\, d\sigma(\xi) \qquad (1)$$

holds. Quantities of the form $(gd\sigma)^\vee$ naturally appear in the study of PDEs. For example, a global solution to the free wave equation

$$\partial_t^2 u(t,x) - \Delta u(t,x) = 0$$

is automatically of the form $u(t,x) = (gd\sigma)^\vee$ where $d\sigma$ denotes the surface measure on the light cone $\{(\tau,\xi) \in \mathbb{R}^{1+d} : |\tau| = |\xi|\}$. Or, a global solution to the free Schrödinger equation

$$i\partial_t u(t,x) + \Delta u(t,x) = 0$$

is of the form $u(t,x) = (gd\sigma)^\vee$ where $d\sigma$ is now the surface measure on the paraboloid $\{(\tau,\xi) \in \mathbb{R}^{1+d} : \tau = |\xi|^2\}$. Thus, the extension estimate measures such as (1) how quickly the solution $(gd\sigma)^\vee$ decays at infinity. The lower the exponent p', the more decay we obtain.

In this lecture we will focus on the relation [4] between the restriction conjecture and a seemingly unrelated problem in geometry, the so-called *Kakeya conjecture*: "If $E \subseteq \mathbb{R}^d$ contains a line segment in every direction, then the Hausdorff dimension of E must be exactly equal to d." This is particularly striking as it is known that such sets – known as Besicovitch sets – can have measure zero!

One way to see that the restriction and Kakeya conjectures are closely related is via the PDE interpretation of the restriction problem. All the above mentioned equations have "wave packet" solutions which resemble a pulse of waves moving in tandem for a short while before dispersing. These can be thought of as oscillatory approximations to straight lines just as the wave equation is an oscillatory approximation to geometric optics. A general solution to such a PDE tends to consist of a superposition of such wave packets, which one can think of as an approximation to a Besicovitch set. The smaller one can make a Besicovitch set, the worse the $L^{p'}$ norm of the solution becomes (i.e., only higher p' are admissible).

1. Remarks on the uncertainty principle, see, e.g., Wolff [18, Chapter 5].

2. Oscillatory integrals of the first kind and decay of Fourier transform of measures supported on hypersurfaces, see, e.g., Stein [11, Chapter XIII].

3. Two necessary conditions and statement of the Fourier restriction conjecture, see, e.g., Wolff [18, Chapter 7].

4. The Tomas–Stein theorem

    (a) Tomas' proof, see, e.g., Wolff [18, Chapter 7] (compact manifolds).

    (b) Stein's proof and complex interpolation [10, 11, 12] (compact manifolds).

    (c) Strichartz estimates [13] (Tomas–Stein for non-compact manifolds) and global well-posedness of $i\partial_t u - \Delta u = \lambda |u|^2 u$ in $d = 2$, see Tao [14, Lecture 4].

    (d) Extension to Schatten ideals, see Frank–Sabin [7, Proposition 1 and Theorem 2].

    (e) Extension to general probability measures, see Bak–Seeger [1].

    (f) Random Tomas–Stein, see Bourgain [5].

    (g) Application in spectral theory (e.g., eigenvalue bounds, Lieb–Thirring).

5. Hausdorff measures, see, e.g., Wolff [18, Chapter 8].

6. First peek at localized restriction and relation to the Kakeya problem. See, e.g., Tao [14, Lecture 5, Section 2] (in particular [15, Theorem 1.2] and Bourgain [4], see also Tao [14, Lecture 7, Proposition 2.2] for two ϵ-removal lemmas) or Hickman–Vitturi [9, Lecture 1, Section 2].

7. First results in the Kakeya problem and how it may help to resolve the restriction conjecture, see Wolff [18, Chapter 10].

8. More on localized restriction, see Tao [14, Lecture 7, Section 2].

9. Wave packet decomposition, see, e.g., Demeter [6, Chapter 2] or Hickman–Vitturi [9, Lecture 1, Section 4].

10. Square function estimates imply the restriction conjecture, see, e.g., Hickman–Vitturi [9, Lecture 1, Section 4].

11. Bilinear restriction, see Tao [14, Lecture 5] or [16, Lecture 1], Hickman–Vitturi [9, Lecture 3], Bennett [2], or Demeter [6, Chapter 3]. See also Tao–Vargas–Vega [17].

12. Multilinear restriction and multilinear Kakeya, see Bennett–Carbery–Tao [3] and Hickman–Vitturi [9, Lecture 3, Sections 2-5].

13. Short proof of multilinear Kakeya, see Guth [8].

References

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