$q<p$: be careful $\qquad r = q-p$

$$g = \sum_m g_m = \sum_m a_m^r \|f_m\|^{p-2} \bar{f}_m \qquad a_m = 2^m W_m^{1/p}$$, f_m quasistep fct of height 2^m and width W_m

$$\sum_m |g_m| \sim 2^{m(q-1)} \cdot W_m^{(q-p)/p} 1_{E_m}$$

~~$\|g\|_{L^{p',q'}}^{q'} \sim \| \|g_m\|_{L^{p'}(\mathbb{R}^d)} \|_{\ell_m^{q'}}^{q'}$ if we knew that the heights of g_m were lacunary.~~ $\cancel{2^{m(q-1)} W_m^{(q-p)/p}}$

$\cancel{\equiv \sum_m}$

Recall Thm 1.2.14 $\|g\|_{L^{p',q'}} \lesssim A \iff \exists$ decomposition $g=\sum g_m$ of quasi-step fcts of heights 2^m, width $\tilde{W}_m$ and the g_m are disjointly supported & $\|2^m \tilde{W}_m^{1/p'}\|_{\ell_m^{q'}} \lesssim A$

This says the following: if we decompose the above g in $g = \sum_m g_m$ where

$$g_m(x) = g(x) \cdot 1_{\{x:\, 2^m \le |g(x)| \le 2^{m+1}\}}, \text{ then}$$

$$\|g\|_{p',q'} \sim \| \|g_m\|_{L^{p'}(\mathbb{R}^d)} \|_{\ell_m^{q'}}$$

$\Rightarrow$ by this observation, $$\|g\|_{p',q'}^{q'} \sim \sum_{A \in 2^{\mathbb{Z}}} A^{q'} \Big(\sum_{m \in M(A)} W_m \Big)^{q'/p'}$$ ~~where $m \in M(A) \iff A \sim 2^{m}$~~

where $m \in M(A) \iff A \sim 2^{m(q-1)} W_m^{\frac{q-p}{p}}$

(fix a dyadic height $A = 2^k$, say, and find out which (of the heights) of the g_m correspond to it and take their $L^{p'}$-norm) which is then given by $A \cdot \|1_{\{|g| \sim A\}}\|_{L^{p'}}$

$$= A \Big\| \sum_{m \in M(A)} 1_{W_m} \Big\|_{p'}$$

$$= A \Big(\sum_{m \in M(A)} W_m \Big)^{1/p'}$$

Now, for each fixed $A \in 2^{\mathbb{Z}}$, the sum in m is over part of a geometric series; indeed,

$$m \in M(A) \iff W_m \sim A^{\frac{p}{q-p}} \cdot 2^{-m\frac{p(q-1)}{q-p}}$$ ($q<p \Rightarrow A$ decreases, but $2^{-mp(q-1)/(q-p)}$ increases)

But for finite ~~subsets~~ subsets $\mathcal{A} \subset 2^{\mathbb{Z}}$ of dyadic (or lacunary) numbers, we have

$$\Big| \sum_{A \in \mathcal{A}} A \Big|^q \le \big| 2 \max_{A \in \mathcal{A}} A \big|^q \le 2^q \sum A^q$$

(think of $\sum_{N \in 2^{\mathbb{Z}},\, N > X} N^{-\alpha} \sim X^{-\alpha}$) or $\sum_{0 < 2^n < X} 2^n \sim X$

$$\Rightarrow \|g\|_{p',q'}^{q'} \sim \sum_{A \in 2^{\mathbb{Z}}} A^{q'} \sum_{m \in M(A)} W_m^{q'/p'}$$

$\frac{1}{q'} = 1 - \frac{1}{q} = \frac{q-1}{q} \iff q'(q-1) = q$

$$\sim \sum_m 2^{mq'(q-1)} W_m^{\frac{q-p}{p} \cdot q'} \cdot W_m^{q'/p'}$$

$\left(\frac{q}{p} - 1\right) \cdot \frac{q}{q-1} + \frac{q/(q-1)}{p/(p-1)} = \dots q/p$

$$= \sum_m \underbrace{2^{mq} W_m^{q/p}}_{a_m^q} \sim 1, \text{ by assumption}$$