(Panorama on conv. of Fourier series) ①

4 Application of the Hilbert transform in convergence of Fourier series

4.1 The "easy" case $d=1$

(Krantz p. 59, 80-84, 87, 92, 121-126, 143-170) Stein - p. 99-103

Let us consider the circle group / torus $\mathbb{R}/\mathbb{Z} \equiv \mathbb{T}$, i.e., the equivalence classes of real numbers that are induced by $x \sim y \Leftrightarrow x = y + h$ for some $h \in \mathbb{Z}$, or, equivalently $\mathbb{T} = [0,1)$ with addition performed modulo 1.

When we think of a function f on $\mathbb{T}$, we naturally identify it with its 2π-periodic extension to the real line.

It's also useful to identify $\mathbb{T}$ with the unit circle S^1 in $\mathbb{C}$ by the mapping

$$\cancel{[0,2\pi) \ni x \mapsto e^{ix} \in S^1} \qquad [0,1) \ni x \mapsto e^{2\pi i x} \in S^1$$

→ fundamental issue in Fourier analysis: does, and in which sense, the formal Fourier series $\sum_{j=-\infty}^{\infty} \hat{f}(j) e^{2\pi i j t} \; (\overset{?}{\sim} f(t))$ converge? Does it even converge to f?

$$\hat{f}(j) = \int_0^1 f(t) e^{-2\pi i j t} dt$$

The question is, under which circumstances do we have a Fourier inversion formula.

<u>Known</u> Riemann-Lebesgue: $f \in L^1(\mathbb{T}) \Rightarrow |\hat{f}(j)| \le \|f\|_{L^1}$ and $\lim_{|j| \to \infty} \hat{f}(j) = 0$

Integration by parts shows that: $f \in C^k(\mathbb{T}) \Rightarrow |\hat{f}(j)| \lesssim_k \langle j \rangle^{-k}$.

To study the convergence of Fourier series, we consider the N-th partial sum

$$(S_N f)(x) := \sum_{j=-N}^{N} \hat{f}(j) e^{2\pi i j x} = \int_0^1 dt \; f(t) \sum_{j=-N}^{N} e^{2\pi i j t} = \int_0^1 dt \; f(t) \cdot D_N(x-t)$$

with $D_N(t) = \dfrac{\sin((2N+1)\pi t)}{\sin(\pi t)}$ (exercise)

D_N: Dirichlet kernel and $\int_0^1 D_N(t)dt = 1$ (using $\int \sum_{j=-N}^{N} e^{2\pi i j t} dt = \int e^{2\pi i 0 \cdot t} dt$)

however $\|D_N\|_{L^1} \sim \log N$

<u>Thm 4.1</u> Let $f \in L^1(\mathbb{T})$ be differentiable at x. Then $(S_N f)(x) \to f(x)$ uniformly, i.e., $|(S_N f)(x) - f(x)| \xrightarrow{N \to \infty} 0$.

<u>Corollary</u> Fourier series of differentiable functions converge to that function at any point! (Contrast this result with the situation for Taylor series)

<u>Proof</u> See Krantz p. 52

<u>Remark</u> In fact, the Dini-type condition $\int_{|t| < 1/2} \frac{|f(t+t_0) - f(t_0)|}{|t|} dt \lesssim 1$ suffices (instead of $f \in C^1$)

Question $S_N f \to f$ pointwise or in L^p, whenever $f \in L^p$?

In Fourier space, S_N corresponds to the rough intervall multiplier

$$m_N(j) = \begin{cases} 1 & \text{if } |j| \le N \\ 0 & \text{if } j > N \end{cases} \equiv a\, \mathbb{1}_{[-N,N]}(j)$$

$$(S_N f)(x) = \sum \hat{f}(j) \cdot m_N(j)\, e^{2\pi i j x}$$

can't be an approximation of the identity! $\Rightarrow$ FAP I cannot be invoked

Since $\|D_N\|_{L^1} \sim \log N$, a direct application of Schur's test doesn't help.

$\to$ exercise using $\left|\frac{1}{\sin t} - \frac{1}{t}\right| \le 1 - \frac{2}{\pi}$ $\forall |t| \le \frac{\pi}{2}$ and $\|D_N\|_{L^1} \sim \sum_{k=1}^{N} \frac{1}{k}$

Lemma 4.2 (weighted Schur)

Let (X,μ), (Y,ν) be measure spaces and $0 \le w(x,y)$ be measurable on $X \times Y$. Suppose the kernel $K(x,y): X \times Y \to \mathbb{C}$ satisfies

$$\sup_{x \in X} \int_Y w(x,y)^{1/p} |K(x,y)|\, d\nu(y) \equiv A_1 < \infty \quad \text{and} \quad \sup_{y \in Y} \int_X w(x,y)^{-1/p'} |K(x,y)|\, d\mu(x) \equiv A_2 < \infty$$

for some $1 < p < \infty$, respectively $1 \le p \le \infty$ when $w(x,y) \equiv 1$. Then, the operator defined by

$$(Tf)(x) = \int_Y K(x,y) f(y)\, d\nu(y) \quad \text{satisfies} \quad \|Tf\|_{L^p(X)} \le A_1^{1/p'} A_2^{1/p} \|f\|_{L^p(Y)}$$

Proof Exercise

We will now show that we nevertheless obtain $S_N f \to f$ in L^p, $1 < p < \infty$.

Theorem 4.3 ($S_N f \to f$ in L^2)

Let $f \in L^2(\mathbb{T})$, then $\|S_N f - f\|_2 \to 0$

Proof Follows from FAP I and the fact that $m_N \in L^\infty$, i.e., by Plancherel, S_N is L^2-bdd. The convergence on dense subsets, such as $C_c^\infty(\mathbb{T})$ is obvious $\square$

Now recall that the Hilbert transform acted as a Fourier multiplier with symbol $-i\,\mathrm{sgn}(\xi)$ in $\mathbb{R}$. Thus,

$$\mathbb{1}_{[-N,N]}(j) = \dots = \frac{1}{2}\left(\mathrm{sgn}(j+N) - \mathrm{sgn}(j-N)\right) + \frac{1}{2}\Big[\underbrace{\mathbb{1}_{\{-N\}}(j)}_{\delta_{-N,j}} + \underbrace{\mathbb{1}_{\{N\}}(j)}_{\delta_{N,j}}\Big]$$

translation $\leftrightarrow$ phase

$$\Rightarrow (S_N f)(e^{2\pi i x}) = \frac{1}{2} i e^{-iNx} H(e_N f)(x) - \frac{1}{2} i e^{iNx} H(e_{-N} f)(x)$$
$$+ \frac{1}{2}\left(P_{-N} f + P_N f\right)(x)$$

where $P_j = |e_j\rangle\langle e_j|$ and $e_j(x) = e^{2\pi i j x}$

To understand this last inequality, let's look, e.g., at

$$\sum_j \mathrm{sgn}(j+N)\hat{f}(j)e^{2\pi i j x} = i\sum_j (-i)\mathrm{sgn}(j)\; e^{-iNx}\; \underbrace{\hat{f}(j-N)}e^{2\pi i j x}$$

$$\int f(t) e^{-2\pi i t(j-N)}\,dt = \int f(t)\cdot e^{2\pi i t N}\cdot e^{-2\pi i t j}\,dt$$

$$= i e^{-iNx}\sum_j -i\,\mathrm{sgn}(j)\cdot (e_N f)^\wedge(j) e^{2\pi i j x}$$

$$= i e^{-iNx}\cdot H(e_N f)(x)$$

Now, we already proved that H is $L^p(\mathbb{R})$-bdd. So, if we knew that the associated convolution operator was also $L^p(\mathbb{T})$-bdd, we would have (by FA? I)

<u>Thm 4.4</u> Let $1<p<\infty$ and $f\in L^p(\mathbb{T})$. Then $\|S_N f - f\|_p \to 0$

The fact that H is also $L^p(\mathbb{T})$-bdd can be deduced from so-called transference principles (see, e.g., Grafakos Section 4.3)

(On $\mathbb{T}$, we have $2(Hf)(x) = \mathrm{P.V.}\int_{-1}^{1} f(x-t)\cot\left(\frac{\pi t}{2}\right)dt$)

However, $\cot\left(\frac{\pi t}{2}\right) - \frac{2}{\pi t} = O(t)$ (exercise!) so the difference between the two kernels is even better than a bounded function and in particular in any L^p

But since we know that p.v. $\frac{1}{x}$ is $L^p(\mathbb{R})$-bdd, we also get that p.v. $\frac{1}{x}$ is $L^p(\mathbb{T})$-bdd by truncating the integral kernel.

In particular, we saw in ex 4.1 that H is not L^1 or L^∞-bdd, so, by FAP I, norm-summability in L^1 or L^∞ (Thm 4.4) fails!

Pointwise convergence of L^p ($1<p<\infty$) is a much deeper problem and was proved by Carleson for $p=2$ and extended by Hunt to general L^p ($1<p<\infty$)

Kolmogorov provided a counterexample and proved pointwise divergence in L^1.

~~In higher dimension~~

4.2 The case $d \geq 2$ – multiple Fourier series

In higher dimensions, matters are not so simple anymore as it's not even clear <u>how</u> we should sum up our Fourier series.

The following notions are somewhat inspired by the classical partial summation operator S_N from dimension 1. But each takes into account the symmetries of space in a different manner, and each has a ~~different~~ distinct analysis.

Let us still stay in $\mathbb{T}^d$, more precisely $\mathbb{T}^2$ for now.

We define, for $f \in L^1(\mathbb{T}^2)$ (i.e. $f(e^{2\pi i s}, e^{2\pi i t}) \in L^1([0,1) \times [0,1))$) the Fourier coefficient

$$\hat{f}(j,k) = \int_0^1 ds \int_0^1 dt \; f(s,t)\, e^{-2\pi i (sj + tk)} \qquad \text{for } j,k \in \mathbb{Z}$$

$$= \int_0^1 ds \int_0^1 dt \; f(s,t)\, e^{-2\pi i (s,t)\cdot(j,k)}$$

Now, the issue is, how do we recover f, ~~to~~ i.e., how to sum the $\hat{f}(j,k)\, e^{2\pi i (jx+ky)}$ up? In the following, we denote by $|(j,k)| = \sqrt{j^2+k^2}$ the euclidean distance.

* ~~Spherical summation~~ $\quad S_R^{sph} f(x,y) = \sum_{|(j,k)|<R} \hat{f}(j,k)\, e^{2\pi i (jx+ky)} \xrightarrow[R\to\infty]{?} f(x,y)$ in which sense?

* ~~Square summability~~ $\quad S_M^{sq} f(x,y) = \sum_{\substack{|j|<M \\ |k|<M}} \hat{f}(j,k)\, e^{2\pi i (jx+ky)} = (D_M^{(2)} * f)(x,y)$

with $D_M^{(2)}(x,y) = \underset{\substack{\uparrow \\ \text{Dirichlet kernel}}}{D_M(x)}\, D_M(y)$

~~Restricted rectangular summability~~ Let $E > 1$ be fixed (eccentricity)

$$S_{(M,N)}^{rect} f(x,y) = \sum_{\substack{|j|<M \\ |k|<N}} \hat{f}(j,k)\, e^{2\pi i (jx+ky)} \quad \text{for } M,N \in \mathbb{N},\ \frac{1}{E} \leq \frac{M}{N} \leq E$$

do we have $\lim_{R\to\infty} \sup_{\substack{M,N>R \\ \frac{1}{E} < \frac{M}{N} < E}} \left| S_{(M,N)}^{rect} f(x,y) - f(x,y) \right| = 0$

pointwise or in L^p norm?

→ "index rectangles" (j,k) must not deviate too much from the square

Unrestricted rectangular summability

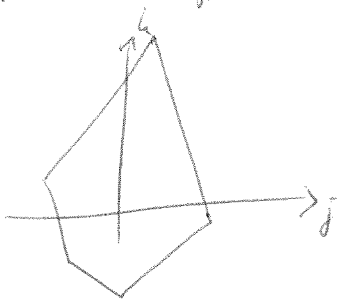
$$\lim_{R\to\infty} \sup_{M,N>R} \left| S^{rect}_{M,N} f(x,y) - f(x,y) \right| \to 0 \quad \text{(pointwise or in norm) ?}$$

↳ here we consider all, arbitrarily eccentric rectangles

Polygonal summability (Cordoba – R. Fefferman)

Fix a closed, convex polygon P with only finitely many sides in the ~~(index) plane~~ (index) plane $\mathbb{Z}^2$. For $R>0$ let $R\cdot P = \{(Rx, Ry) : (x,y)\in P\}$ the R-dilated polygon.

$$S_R^{poly\,P} f(x,y) = \sum_{(j,k)\in R\cdot P} \hat{f}(j,k)\, e^{2\pi i (jx+ky)} \xrightarrow[R\to\infty]{?} f$$



Clearly, square partial summation is a special case of polygonal partial summation although they are quite similar (polygonal summation is just a larger linear combination of Hilbert transforms)

However, unrestricted rectangular and spherical summation stand far apart.

lack of control of eccentricity!

tangent lines to the boundary of the ball point in too many directions!

4.3 Transference principles

In the following, we will argue (without proof though!) that it suffices to prove $L^p(\mathbb{R}^d)$ bddness of certain Fourier multipliers to obtain summability of Fourier series. The machinery that makes all this work is called transference and is dealt with in detail in Grafakos Sections 4.3. (4.3.1 – 4.3.3)

or Krantz Section 3.3

Remark (Thm 4.3.1 in Grafakos) A linear operator commuting with translations mapping $L^p(\mathbb{T}^d)$ to $L^q(\mathbb{T}^d)$ $(1\le p,q\le\infty)$ can be represented as

$$(Tf)(x) = \sum_m a_m \hat{f}(m) e^{2\pi i m x} \quad \text{for some bdd sequence } (a_m)_{m\in\mathbb{Z}^d} \text{ with}$$

$$\|(a_m)_m\|_{\ell^\infty} \le \|T\|_{L^p\to L^q}.$$

Prop 3.5, 3.6 notes and Thms 2.5.2, 2.5.8 in Grafakos

Lemma 4.6 Let f be a continuous function on $\mathbb{R}^d$ and 1-periodic in each variable separately.

Then $\lim_{\epsilon\to 0} \epsilon^{d/2} \int_{\mathbb{R}^d} f(x) e^{-\epsilon\pi|x|^2/2} dx = \int_{Q_d} f(x)dx$ where $Q_d = (-1,1)^d$

$\longrightarrow$ exercise

Lemma 4.7 Let P, Q be trigonometric polynomials (i.e., of the form $\sum_{j=-N}^{N} a_j e^{2\pi i j t}$; these are dense in $L^p(\mathbb{T})$, $1\le p<\infty$)

Let S be transl. inv. $L^p(\mathbb{R}^d)$-bdd for $1<p<2$ with associated Fourier multiplier s.

Assume that the function s is continuous at each $j\in\mathbb{Z}^d$.

Suppose, $f(x) = \sum a_j e^{2\pi i j x}$ is a trigonometric polynomial and set

$$(\tilde S f)(x) := \sum_{j\in\mathbb{Z}^d} s(-j)\, a_j\, e^{2\pi i j\cdot x}$$

Let $\alpha,\beta>0$ satisfy $\alpha+\beta=1$. Then, for $\eta_{\sqrt{\epsilon}}(x) = \epsilon^{d/2} e^{-\epsilon\pi|x|^2/2}$

$$\lim_{\epsilon\to 0}\left(\frac{1}{\alpha\beta\epsilon}\right)^{d/2}\int_{\mathbb{R}^d}\left[S(P\cdot\eta_{\sqrt{\epsilon\alpha}})(x)\right]\left[\overline{Q(x)\cdot\eta_{\sqrt{\epsilon\beta}}(x)}\right]dx = \int_{Q_d}\left[(\tilde S P)(x)\right]\left[\overline{Q(x)}\right]dx$$

Theorem 4.8 (see also Grafakos Thm 4.3.7) Let $1<p<\infty$ and suppose S is a transl. inv. $L^p(\mathbb{R}^d)$-bdd ~~multiplier~~ operator with associated multiplier s. Assume that s is continuous at each point in $\mathbb{Z}^d$. Then there is a unique periodized operator $\tilde S$ given by

$$\tilde S f \sim \sum_{j\in\mathbb{Z}^d} s(-j)\hat f(j) e^{2\pi i j\cdot x}, \quad f\in L^p(\mathbb{T}^d)$$

s.t. $\tilde S$ is a multiplier in $L^p(\mathbb{T}^d)$ with $\|\tilde S\|_{L^p(\mathbb{T}^d)\to L^p(\mathbb{T}^d)} \le \|S\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)}$

Some technical modifications of the proof of Thm 4.7 give the following useful

Corollary 4.8' Thm 4.8 holds under the mere assumption

$$\forall j\in\mathbb{Z}^d: \lim_{\epsilon\to 0} \epsilon^{-d}\int_{|t|<\epsilon} dt\,|s(j-t)-s(j)| = 0 \quad \text{(Dini)}$$

I.e., Thm 4.8 holds as long as each $j\in\mathbb{Z}^d$ is a Lebesgue point for the Fourier multiplier s (e.g., the case for balls or cubes!)

$\rightarrow$ The above thm thus gives us a way of passing from a Fourier multiplier on $\mathbb{R}^d$ to a multiplier for Fourier series, and hence a summation method for multiple Fourier series.

Although, we won't need it, we state the following converse (without proof, but see Grafakos Thm 4.3.10.

<u>Theorem 4.9</u> Suppose s is a continuous, complex-valued function on $\mathbb{R}^d$ and assume that for each $\epsilon > 0$, the operators given by

$\tilde{S}_\epsilon f(x) \sim \sum_{j \in \mathbb{Z}^d} s(-\epsilon j)\, \hat{f}(j)\, e^{2\pi i j \cdot x}$ are uniformly (in ϵ) $L^p(\mathbb{T}^d)$-bdd for $1 < p < \infty$.

Then s is a Fourier multiplier on $\mathbb{R}^d$, ~~and~~ with $\|M_s\|_{L^p(\mathbb{R}^d)}$

$$\|M_s\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \le \sup_{\epsilon > 0} \|\tilde{S}_\epsilon\|$$

(linear operator on $L^p(\mathbb{R}^d)$ corresponding to the Fourier multiplier s.

($\to$ another transference principle of de Leeuw for transferring multiplier bounds from $\mathbb{R}^d$ to $\mathbb{R}^{d-1}$ will be discussed later in Fefferman's disproof of the ball multiplier conjecture)

<u>4.4 Applications of Fourier multiplier theorems (via transference) to summation of multiple Fourier series</u>

(Krantz, Sect 3.4
Stein, Ch IV, §4

We will now apply Thms 4.8, 4.9 to the problem of summing multiple Fourier series wher we will only be concerned with norm convergence

We begin with square / cubical summation.

Let $Q = [-1, 1]^d$ be the cube centered at 0 with side length 2

$s := \mathbb{1}_Q$ the associated Fourier multiplier.

$1 < p < \infty$

We would like to apply Thm 4.8 which we can, however, <u>not</u> directly apply because $\mathbb{1}_Q$ is not continuous at each lattice point. The matter can be handled in ~~two ways~~ the following way (there is another $\to$ p. 141 in Krantz)

(i) dilate Q just a little, so that the dilate, say $\tilde{Q}$, so that any lattice point is either strictly <u>interior to the open cube $\tilde{Q}$</u> <u>or</u> strictly <u>exterior to the closure $\overline{\tilde{Q}}$</u>. In this case, the hypotheses of Corollary 4.8' are certainly satisfied $\Rightarrow f \mapsto \sum_{j \in \mathbb{Z}^d} \mathbb{1}_{\tilde{Q}}(-j)\, \hat{f}(j)\, e^{2\pi i j \cdot x}$ is ~~L^p-bdd~~ $L^p(\mathbb{T}^d)$-bdd and the bound can be taken to be independent of the infinitesimally small perturbation $\tilde{Q}$ of Q. $\Rightarrow f \mapsto \sum_{j \in \mathbb{Z}^d} \mathbb{1}_Q(-j)\, \hat{f}(j)\, e^{2\pi i j \cdot x}$ $L^p(\mathbb{T}^d)$-bdd, too.

~~(ii) Invoke Corollary 4.8' directly~~

Lemma 4.10 Let $1<p<\infty$ and Q_R be the R-fold dilate of Q. If the multiplier M_{1_Q} is $L^p(\mathbb{R}^d)$-bdd, then

$M_{1_{Q_R}} : f \mapsto (1_{Q_R}\hat{f})^\vee$ is $L^p(\mathbb{R}^d)$-bdd with norm equal to the

$L^p \to L^p$-bound of M_{1_Q}

→ by Thm 4.8 + Lemma 4.10

Corollary 4.11 If M_{1_Q} is $L^p(\mathbb{R}^d)$-bdd for some $p\in(1,\infty)$, then the periodized operator $f \mapsto \sum_{j\in\mathbb{Z}^d} 1_{Q_R}(-j)\hat{f}(j)e^{2\pi i j\cdot x}$ is $L^p(\mathbb{T}^d)$-bdd with operator norm independent of R. In particular, we have $L^p(\mathbb{T}^d)$-convergence of cubical summation.

~~Proof of Lemma 4.10 By a density argument (at the end) it suffices to consider $f\in S$~~

~~The rest~~

Proof of Lemma 4.10 as in Ex 7.3 or 8.1.

Summarized $L^p(\mathbb{R}^d)$-bddness of cube multiplier $\Rightarrow$ cubic L^p-summability of Fourier series.

By Thm 4.9, the converse holds, too. (again, the hypotheses can be weakened to supposing that each lattice point is a Lebesgue point for 1_Q; just as we have discussed in connection with Corollary 4.8'.

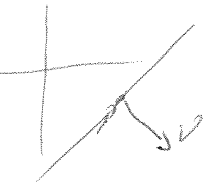
Thus, we're left with proving $L^p(\mathbb{R}^d)$-bddness of the cube multiplier. ($\sim$ Hilbert transform, as we'll shortly see). For convenience, we'll just consider $d=2$. Recall $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$.

$$Hf(x) = \text{p.v. } \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x-y)}{y}\,dy$$

$H: L^p(\mathbb{R}) \to L^p(\mathbb{R})$ bdd $1<p<\infty$ and weak-type $(1,1)$

Using the L^p-bddness of H, we immediately obtain

Theorem 4.12 Let $P\in\mathbb{R}^2$, $\nu\in S^2$ a unit vector, and $E_\nu=\{x\in\mathbb{R}^2 : (x-P)\cdot\nu\geq 0\}$ a half plane beginning at P, oriented along ν. Then $f\mapsto(\mathbb{1}_{H_\nu}\hat{f})^\vee$ is $L^p(\mathbb{R}^2)$-bdd for $1<p<\infty$.



Proof: Let $1<p<\infty$. By a density argument (at the end), we'll merely consider $f\in\mathcal{S}(\mathbb{R}^2)$ and Since the multiplier of H is $-i\,\mathrm{sgn}\,\xi$, we can write

$$\mathbb{1}_{[0,\infty)}(\xi)=\frac{1}{2}\left(1+i\cdot(-i\,\mathrm{sgn}\,\xi)\right) \quad \text{a.e.},$$

we have $(\mathbb{1}_{H_\nu}\hat{f})^\vee=\frac{1}{2}(1+iH)f$.

We now express the half-plane multiplier as an amalgam of half-line multipliers (using Fubini). By a translation and rotation, it suffices to consider $E_{(0,1)}=\{x\in\mathbb{R}^2 : x\cdot\binom{0}{1}\geq 0\}\equiv E$.

Recall $1<p<\infty$, $f\in\mathcal{S}(\mathbb{R}^2)$. For almost every $x_1\in\mathbb{R}$, the function $f_{x_1}(x_2)\equiv f(x_1,x_2)$ is in $L^p_{x_2}(\mathbb{R}^1)$ and, by Fubini, $\int_{\mathbb{R}}\|f_{x_1}\|^p_{L^p_{x_2}(\mathbb{R})}=\|f\|^p_{L^p(\mathbb{R}^2)}$. Next, we may write

$$(\mathbb{1}_E\hat{f})^\vee(x_1,x_2)=\int_0^\infty d\xi_2\int_{\mathbb{R}}d\xi_1\int_{\mathbb{R}^2}dy_1\,dy_2\; f(y_1,y_2)\,e^{2\pi i\xi_1(y_1-x_1)}\,e^{2\pi i\xi_2(y_2-x_2)}$$

$$\overset{\text{Fubini}}{=}\int_{\mathbb{R}}d\xi_1\int_{\mathbb{R}}dy_1\; e^{2\pi i\xi_1(y_1-x_1)}\int_0^\infty d\xi_2\int_{\mathbb{R}}dy_2\; f(y_1,y_2)\,e^{2\pi i\xi_2(y_2-x_2)}$$

$$=\int_{\mathbb{R}}d\xi_1\int_{\mathbb{R}}dy_1\; e^{2\pi i\xi_1(y_1-x_1)}\underbrace{(\mathbb{1}_{[0,\infty)}\hat{f}_{y_1})^\vee(x_2)} \qquad (**)$$

meaningful $\forall\, y_1$ as $f_{y_1}\in L^p$ and in $L^p_{x_2}$. Moreover,

~~Moreover~~

$$\int\|(\mathbb{1}_{[0,\infty)}\hat{f}_{y_1})^\vee\|^p_{L^p_{x_2}}\,dy_1 \lesssim \int\|f_{y_1}(x_2)\|^p_{L^p_{x_2}}\,dy_1=\|f\|_p^p \quad (*)$$

$\Rightarrow$ $(\mathbb{1}_{[0,\infty)}\hat{f}_{y_1})^\vee(x_2)$ is ~~in $L(\mathbb{R}^2)$~~ in $L^p_{y_1,x_2}(\mathbb{R}^2)$. In particular, for a.e. $x_2\in\mathbb{R}$, the function $y_1\mapsto F_{x_2}(y_1)=(\mathbb{1}_{[0,\infty)}\hat{f}_{y_1})^\vee(x_2)\in L^p_{y_1}(\mathbb{R})$

Moreover, by $(*)$,

$$\int_{\mathbb{R}} dx_2 \int_{\mathbb{R}} dx_2 \, \| F_{x_2}(y_1) \|^p_{L^p_{y_1}(\mathbb{R})} = \int_{\mathbb{R}} dy_1 \, \| (\mathbb{1}_{[0,\infty)} \hat{f}_{y_1})^\vee (x_2) \|^p_{L^p_{x_2}(\mathbb{R})} \lesssim \|f\|^p_{L^p(\mathbb{R}^2)}$$

$\Rightarrow$ We may rewrite RHS of $(**)$ as $\lim\limits_{\epsilon \to 0} \int_{\mathbb{R}} d\xi_1 \int_{\mathbb{R}} dy_1 \, F_{x_2}(y_1) \, e^{2\pi i \xi_1 (y_1 - x_1)} e^{-\epsilon |\xi_1|^2}$

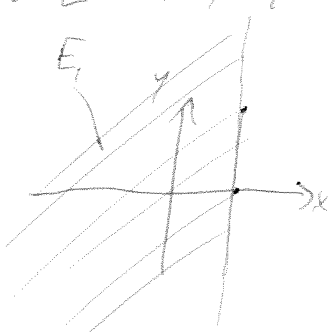
(by "Gauss-Weierstraß summation" ($\to$ Thm 2.6 for $F_{x_2}(y_1) \in L^p_{y_1 x_2}(\mathbb{R}^2)$))

which equals $\lim\limits_{\epsilon \to 0} \int_{\mathbb{R}} d\xi_1 \, e^{-\epsilon |\xi_1|^2} e^{-2\pi i \xi_1 \cdot x_1} \check{F}_{x_2}(\xi_1) = F_{x_2}(x_1)$ for a.e. $x_1 \in \mathbb{R}$

But since we've already seen that $F_{x_2}(x_1) \in L^p_{x_1, x_2}(\mathbb{R}^2)$ $\left(\| F_{x_2}(x_1) \|_{L^p_{x_1 x_2}(\mathbb{R}^2)} \lesssim \|f\|_{L^p} \right)$

the proof of the theorem is complete. $\boxtimes$

Thm 4.13 We have square-summability of Fourier series in $L^p(\mathbb{R}^2)$, $1<p<\infty$.



Proof Let $E_1 = \{ (x,y) \in \mathbb{R}^2 : (-1, 0) \cdot [(x,y) - (1,0)] \geq 0 \}$

$E_2 = \{ (x,y) \in \mathbb{R}^2 : (1, 0) \cdot [(x,y) - (-1,0)] \geq 0 \}$

$E_3 = \{ (x,y) \in \mathbb{R}^2 : (0,-1) \cdot [(x,y) - (0,1)] \geq 0 \}$

$E_4 = \{ (x,y) \in \mathbb{R}^2 : (0,1) \cdot [(x,y) - (0,-1)] \geq 0 \}$

be four half-planes whose common intersection is the 2×2 square $Q = \{ (x,y) : |x|, |y| \leq 1 \} \in \mathbb{R}^2$. Since the associated half-plane multipliers $\mathbb{1}_{E_j}$ are all $L^p(\mathbb{R}^2)$-bdd, so is $T_1 \circ T_2 \circ T_3 \circ T_4$.

$(T_j f(x) = (\mathbb{1}_{E_j} \hat{f})^\vee (x))$ $\Rightarrow$ By Corollary 4.11, we obtain the desired conclusion.

$\boxtimes$

By similar arguments, one obtains square summability for Fourier series in $L^p(\mathbb{T}^d)$ for any $d = 2, 3, \ldots$. Moreover, the arguments can be generalized to obtain polygonal summability. $\to$ next page

Thm 4.14 (Polygonal summability on $\mathbb{T}^d$)

Let P be any closed, convex polygon (convex polyhedron) in $\mathbb{R}^2$ (resp. $\mathbb{R}^d$) with non-empty interior. Then the Fourier multiplier $\mathbb{1}_P$ is $L^p(\mathbb{R}^d)$-bdd for $1<p<\infty$. In particular, polygonal summation wrt this P is valid in L^p $(1<p<\infty)$.

Restricted rectangular convergence is a lot trickier than that.

Example: polygonal pointwise-a.e. convergence holds, but fails for restricted rectangular convergence

(Fefferman - On the convergence of Fourier series '71) (Bulletin of AMS)

(Fefferman - On the divergence of Fourier series '71) (Bulletin of AMS)

As we'll see next, the arguments presented cannot be good enough to get spherical summation (in $d=2$: disc = intersection of infinitely many half-planes)
→ bound is only likely to blow up; it may just be that the method of our proofs is just not good enough...
However, what seems to be going wrong with the suggested proof above is intrinsic. The disc multiplier is only L^p-bdd for $p=2$!
We will discuss this at the end of these lectures!

Remark: From the proof it follows that, for an arbitrary rectangle ρ in $\mathbb{R}^d$ (cartesian product of d intervals)

we have $\|S_\rho f\|_{L^p} \le A_p \|f\|_p$, $1<p<\infty$, $f \in L^2 \cap L^p$ (independent of rectangle)

→ Stein, Ch IV, §4, Thm 4

(What we've proved so far is only an d-fold superposition of a one-dimensional result, i.e., not genuinely a d-dimensional result)

~~4.5 Hörmander-Mikhlin multiplier theorem~~

4.5 Ball multiplier problem

In this section we will see that our current understanding of euclidean harmonic analysis reveals that virtually any deep question about multidimensional harmonic analysis deeply relates to fundamental problems/ideas about euclidean geometry → Kakeya's needle problem

Problem Dip an infinitely thin needle (a unit-line segment in $d=2$ if you wish) into a bottle of ink and place it on a piece of paper.

→ Challenge: move the needle on the piece of paper as to

(i) reverse the positions of the two ends &

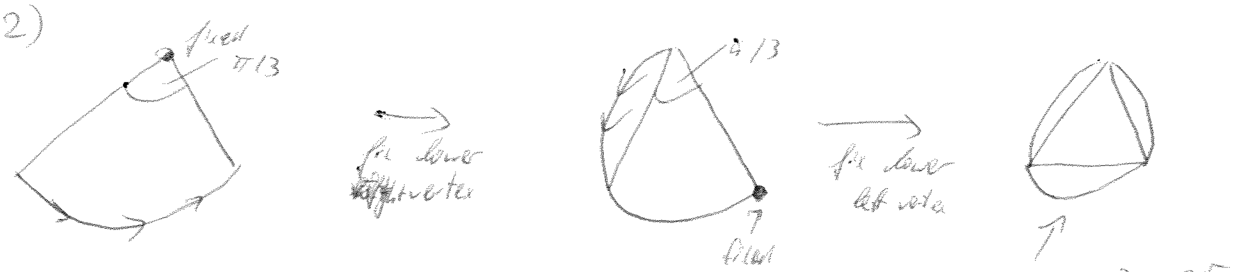
(ii) leave behind an ink blot of smallest possible area.

Mathematically Find the smallest set (and compute its measure) that contains a unit line segment in every direction.

Kakeya set: set that contains a unit line segment in every direction

Easy examples: 1) Rotate the needle by 2π ⇒ circle of radius 1 with area π

2)

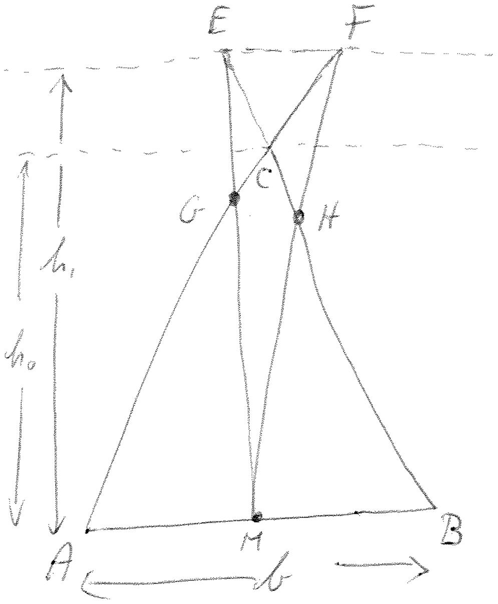


Shockingly, Besicovitch (1926) actually proved that there is no lower bound >0 for the area of Kakeya sets. More precisely: for every $\epsilon > 0$ there is $E \subseteq \mathbb{R}^2$ such that $|E| < \epsilon$ and a needle in E may have its ends switched in such a way that the needle remains in E at all times

Easier constructions were given later by O. Perron (1927) (Perron trees) and further simplified by J. Schoenberg (1960s). See also Cunningham 1970s

Modern Version of the problem: Do Kakeya sets still have Hausdorff dimension $=d$?
(Question settled in $d=2$, $d\geq 3$ open!)

4.5.1 Sprouting of triangles



- Given a triangle ABC with base $b = AB$ and height h_0, let M be the midpoint of AB
- Now fix a height $h_1 > h_0$ and extend the sides $AC \to AF$ and $BG \to BE$.
- AMF and BME are called <u>sprouts</u> of ABC
- $AMF \cup BME$ is called <u>sprouted figure obtained from ABC</u> and is denoted by <u>spr(ABC)</u>; clearly, we have spr $ABC \supseteq ABC$
- We call the set difference <u>spr $ABC \setminus ABC$ the ears of the sprouted figure</u>, i.e., $EGC \cup FCH$.

<u>Exercise</u>: $\text{area}(EGC) = \frac{b}{2} \frac{(h_1 - h_0)^2}{2h_1 - h_0}$

need to turn a pole of length 1 inside this arena!

<u>Goal</u> Scale $b \mapsto \epsilon b$, i.e., the lengt scale of the problem $\sim \epsilon \to$ expected to have area $\sim \epsilon^2$

(or $b \mapsto Nb$; what one can actually show is that the area of Kakeya sets is roughly N! By scaling back, this gives Besicovitch's theorem).
($\to$ turn around a pole of lengt N around this arena)

<u>Concretely</u> Consider an isosceles triangle ~~AB~~ $\Lambda \equiv ABC$ in $\mathbb{R}^2$ with base AB of length <u>$b_0 = \epsilon$</u> and height <u>$h_0 = MC = \epsilon$</u>. Now define the sequence of heights

$\sim \frac{\epsilon^2}{2}$

$h_1 = (1 + \frac{1}{2})\epsilon$

$h_2 = (1 + \frac{1}{2} + \frac{1}{3})\epsilon$

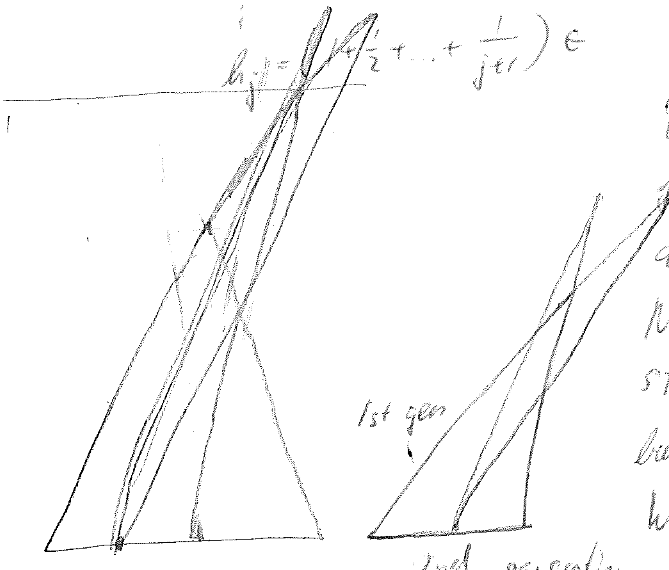
$\vdots$

$h_j = (1 + \frac{1}{2} + \dots + \frac{1}{j+1})\epsilon$

and apply the previously described sprouting procedure to $\Lambda \equiv ABC$ to obtain two sprouts $\Lambda_1 = AMF$ and $\Lambda_2 = BME$ with heights h_1 and base length $b_1 = b_0/2$. Now apply this procedure to Λ_1, Λ_2 to obtain sprouts $\Lambda_{11}, \Lambda_{12}$ from Λ_1 and $\Lambda_{21}, \Lambda_{22}$ from Λ_2, i.e., four sprouts at height h_2. ~~Iterate~~

Now iterate this procedure to obtain, at the j-th step, 2^j sprouts $\Lambda_{r_1 \dots r_j}$, $r_1, \dots, r_j \in \{1,2\}$ each with base length $b_j = 2^{-j} b_0$ and height h_j.

We stop this process at a certain k.



We denote by $E(\epsilon, h) = \bigcup_{j=0}^{h} \bigcup_{\tau_1,\dots,\tau_j=1,2} \Lambda_{\tau_1,\tau_2,\dots,\tau_j}$

We can bound $|E(\epsilon,h)|$ from above by adding to $|\Lambda|$ the area of all the ears of the sprouted figures obtained from the construction. By the exercise on p. 13, ~~the~~ we have

$$|ear_{j\text{-th step}}| = \frac{b_{j-1}}{2} \frac{(h_j - h_{j-1})^2}{2h_j - h_{j-1}}$$

→ Summing over all areas, we obtain

$$|E(\epsilon,h)| = \frac{\epsilon^2}{2} + \sum_{j=1}^{h} \underbrace{2^j}_{\text{number of ears at step } j} \cdot \frac{b_{j-1}}{2} \frac{(h_{j} - h_{j-1})^2}{2h_j - h_{j-1}}$$

$2h_j - h_{j-1} \geq \epsilon$

$$\leq \frac{\epsilon^2}{2} + \sum_{j=1}^{h} 2^j \cdot \frac{2^{-(j-1)} b_0}{2} \frac{\epsilon^2}{(j+1)^2 \epsilon}$$

$$< \frac{\epsilon^2}{2} + \sum_{j=2}^{\infty} \frac{\epsilon^2}{j^2} = \left(\frac{1}{2} + \frac{\pi^2}{6} - 1\right)\epsilon^2 \leq \frac{3}{2}\epsilon^2$$

We indicate some idea that appear in the solution of the <u>classical</u> Kakeya problem

1) No matter what h is, the measure $|E(\epsilon,h)|$ can be made as small as we please by taking $\epsilon \to 0$
2) <u>Goal</u> (recall) move a unit line segment from one side of this angle to the other utilizing each sprouted triangle in succession.
   To achieve this, we need to apply a similar construction to any of the 2^h triangles that make up the set $E(\epsilon,h)$ and repeat the sprouting a large number of times
3) An elaborate construction of this sort yields a set within which the needle can be turned only through a fixed angle. By adjoining a few of such sets, one can rotate the needle through a half-turn within a set that still has arbitrarily small area

4.5.2 The counterexample

Recall exercise 8.3, i.e.,

Lemma 4.15 (Y. Meyer)

Let $v_1, v_2, \dots, v_j, \dots \in S^1$ and $H_j = \{x \in \mathbb{R}^2 : x \cdot v_j \geq 0\}$ be a half-plane oriented along the direction v_j. Let $(S_{H_j} f)(x) = (\mathbb{1}_{H_j} \hat{f})(x)$ be the associated half-plane multipliers. Assume that the disc multiplier $Tf = (\mathbb{1}_{B_0(1)} \hat{f})^\vee$ maps $L^p(\mathbb{R}^2)$ to itself with norm $B_p < \infty$. Then the square function estimate

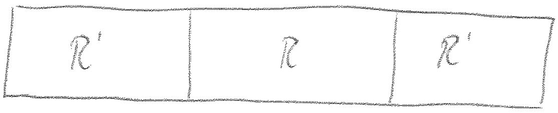
$$\Big\| \Big(\sum_j |S_{H_j} f_j|^2 \Big)^{1/2} \Big\|_p \leq B_p \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_p, \quad f_j \in L^p(\mathbb{R}^2)$$

holds.

Our goal is to prove that the above square function can, in fact, never hold, unless $p = 2$.

Our example is based on a slight variant of (Schönberg's improvement of) Besicovitch's solution of the Kakeya needle problem.

Notation.



R … given rectangle

R' … two copies of R, glued to the short side of R

Lemma 4.16 (Schönberg's construction)

Let $\delta > 0$ be a given number. Then there exist $E \subseteq \mathbb{R}^2$ measurable and a finite collection of rectangles $R_j \subseteq \mathbb{R}^2$ such that

(i) the R_j's are pairwise disjoint

(ii) we have $1/2 \leq |E| \leq 3/2$

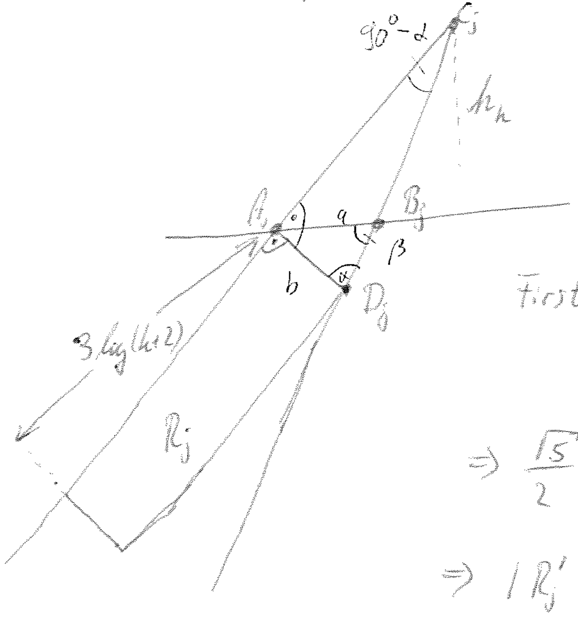
(iii) we have $|E| \leq \delta \sum_j |R_j|$

(iv) for all j we have $|R_j' \cap E| \geq \frac{1}{12} |R_j|$.

Proof Let $A=(0,0)$, $B=(1,0)$ and let C be such that ABC is an isosceles triangle with height 1. Given δ, let $k\in\mathbb{N}$ be such that $k+2>e^{1/\delta}$. For this k we set $E=E(1,k)$, the Besicovitch set from the previous subsection with $\epsilon=1 \Rightarrow \frac{1}{2}\le |E|\le 3/2$, i.e. (ii) holds.

Recall that any dyadic interval $[j\cdot 2^{-k}, (j+1)2^{-k}]\subseteq[0,1]$ is the base of exactly one sprouted triangle $A_jB_jC_j$, where $j\in\{0,1,\dots,2^k-1\}$, $A_j=(j2^{-k},0)$, $B_j=((j+1)2^{-k},0)$ and C_j the top vertex of the sprouted triangle of height $h_k=\sum_{l=1}^{k+1}\frac{1}{l}\in(\log(k+2),\ 1+\log(k+1))\subseteq(\log(k+2),\ 2\log(k+2))$ $(*)$

Next, we find our rectangles R_j. We zoom in to get a closer look at $A_jB_jC_j$



R_j ... rectangle with one vertex either A_j or B_j and side length $3\log(k+2)$.

We will now prove $|R_j\cap E|>\frac{|R_j|}{12}$

First: $\max\{A_jC_j, B_jC_j\}\le\frac{\sqrt{5}\,h_k}{2}<\frac{3h_k}{2}$; by symmetry, let's assume wlog $A_jC_j\ge B_jC_j$.

$\Rightarrow \frac{\sqrt{5}}{2}h_k<\frac{3}{2}h_k\overset{(*)}{<}3\log(k+2)\Rightarrow \overset{\text{right upper copy}}{R_j'}\supseteq A_jB_jC_j$

$\Rightarrow |R_j'\cap E|\ge|A_jB_jC_j|=\frac{1}{2}\cdot 2^{-k}\cdot h_k\overset{(*)}{>}2^{-k-1}\log(k+2)$ $(**)$

Notation $X,Y\in\mathbb{R}^2$: let $\overrightarrow{XY}$ denote the line segment through X and Y

$\rightarrow$ apply law of sines to $A_jB_jD_j \Rightarrow \frac{2^{-k}}{|A_jD_j|}=\frac{\sin\measuredangle A_jD_jB_j}{\sin\measuredangle A_jB_jD_j}$

$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$

$\Rightarrow |A_jD_j|=2^{-k}\frac{\sin\measuredangle A_jB_jD_j}{\sin\measuredangle A_jD_jB_j}=2^{-k}\frac{\sin\measuredangle A_jB_jD_j}{\cos\measuredangle A_jC_jB_j}$

$\le\frac{2^{-k}}{\cos\measuredangle A_jC_jB_j}$

$c^2=a^2+b^2-2ab\cos\gamma$

But the law of cosines applied to $A_jB_jC_j$ shows $\quad h_k\ge\log(k+2)>2^{-k-1}\ (k\ge1)$

$$\cos(\measuredangle A_jC_jB_j)=\frac{|A_jC_j|^2+|B_jC_j|^2-|A_jB_j|^2}{2|A_jC_j||B_jC_j|}\ge\frac{2h_k^2-(2^{-k})^2}{2\cdot\frac{5}{4}\cdot h_k^2}\ge\frac{4}{5}-\frac{2}{5}\cdot\frac{1}{4}\ge\frac{1}{2}$$

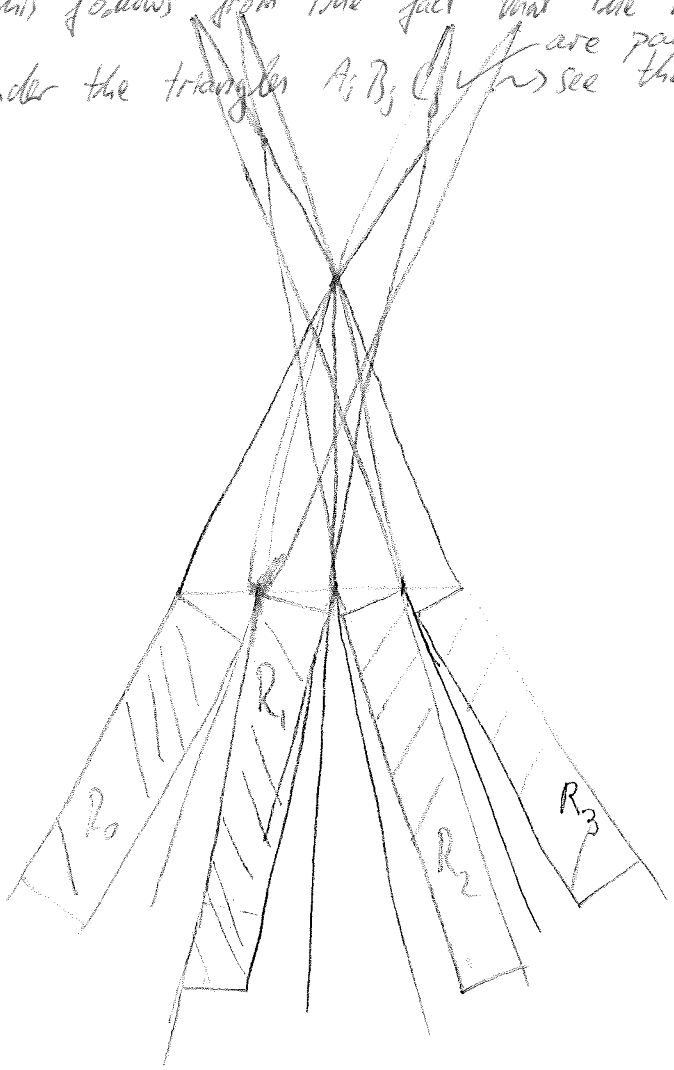
$h_k\le|A_jC_j|,|B_jC_j|\le\sqrt{5}\,h_k/2$

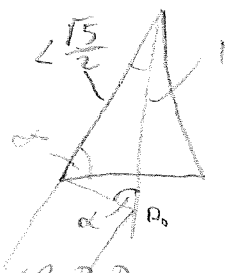
Combining this with $|A_jD_j|\le\frac{2^{-k}}{\cos\measuredangle A_jC_jB_j}$ yields $|A_jD_j|\le 2^{-k+1}=2|A_jB_j|$

$\Rightarrow$ By $|R_j'\cap E|\overset{(**)}{>}2^{-k-1}\log(k+2)=\frac{1}{12}2^{-k+1}\cdot 3\log(k+2)\ge\frac{1}{12}|R_j| \Rightarrow$ (iv)

We will now show (i), i.e., the R_j's are pairwise disjoint.

This follows from the fact that the regions inside the angles $\measuredangle A_j C_j B_j$ and under the triangles $A_j B_j C_j$ are pairwise disjoint $\rightsquigarrow$ see the following picture





Remains to prove (ii), i.e., $|E| \leq \delta \sum_j |R_j|$

By the law of sines, $\dfrac{|A_j D_j|}{\sin \measuredangle A_j B_j D_j} = \dfrac{2^{-k}}{\sin \measuredangle A_j D_j B_j} \Rightarrow |A_j D_j| \geq 2^{-k} \underbrace{\sin \measuredangle A_j B_j D_j}_{= \cos A_j D_j B_j}$

$$\geq 2^{-k} \cos \measuredangle A_0 B_0 B_0$$

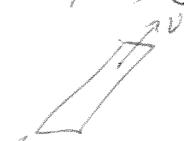
$$= 2^{-k} \cdot \frac{\sqrt{5}}{2 \cdot 2} > 2^{-k-1}$$

$\Rightarrow \text{area}(R_j) \geq 2^{-k-1} \cdot 3 \log(k+2)$

$\Rightarrow \left| \bigcup_{j=0}^{2^k-1} R_j \right| \underset{\substack{\uparrow \\ R_j \text{ disjoint (i)}}}{=} \sum_{j=0}^{2^k-1} |R_j| \geq 2^k \cdot 2^{-k-1} \cdot 3 \log(k+2) \underset{|E| < 3/2}{\geq} |E| \log(k+2) \underset{\text{choice } k+2 > e^{1/\delta}}{\geq} \dfrac{|E|}{\delta}$

$\blacksquare$

Proposition 4.17 (Pointwise lower bound on rectangle multiplier in x-space)



Let $R \subseteq \mathbb{R}^2$ be a rectangle centered at $\vec{0}$, oriented along $v \in S^1$.

Let $H = \{ x \in \mathbb{R}^2 : x \cdot v \geq 0 \}$ be the correspondingly oriented half-plane and $S_H f = (\hat{f} \mathbb{1}_H)^\vee$ be the associated half-plane multiplier.

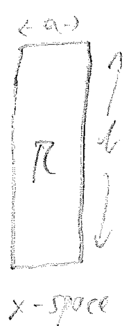
Then, $|S_H(\mathbb{1}_R)(x)| \geq \frac{1}{10} \mathbb{1}_{R'}(x)$.

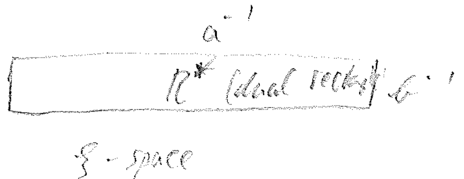
Remark Applying translations, one sees that the same conclusion holds for any rectangle in $\mathbb{R}^2$ oriented ~~alg~~ along v.

Proof Applying a rotation, we can assume wlog $R = [-a,a] \times [-b,b]$ with $0 < a \leq b < \infty$ and $v = e_2 = (0,1)$. ~~In~~ Since FT acts in each variable separately, we have

$$S_H(\mathbb{1}_R)(x_1,x_2) = \mathbb{1}_{[-a,a]}(x_1) \left(\widehat{\mathbb{1}_{[-b,b]}} \, \mathbb{1}_{[0,\infty)} \right)^\vee (x_2)$$

$$= \mathbb{1}_{[-a,a]}(x_1) \, \frac{1 + i\underbrace{H}_{\text{Hilbert transform}}}{2} \, \mathbb{1}_{[-b,b]}(x_2)$$





$\widehat{\mathbb{1}_R} \sim \mathbb{1}_{R^*}$ χ_{R^*}

not Schwartz, but decays like $\langle \xi \rangle^{-\frac{d+1}{2}}$

$|a + ib| > |b|$

$$\Rightarrow |S_H(\mathbb{1}_R)(x_1,x_2)| \geq \frac{1}{2} \mathbb{1}_{[-a,a]}(x_1) \, |H(\mathbb{1}_{[-b,b]})(x_2)| \overset{\text{exercise 5.2}}{=} \frac{1}{2\pi} \mathbb{1}_{[-a,a]}(x_1) \left| \log \left| \frac{x_2+b}{x_2-b} \right| \right|$$

Now, for $(x_1,x_2) \in R'$, we have $\mathbb{1}_{[-a,a]}(x_1) = 1$ and $b < |x_2| < 3b$ ~~to~~

$= [-a,a] \times [b, 3b] \cup [-a,a] \times [-3b,-b]$

$\Rightarrow$ two cases. $b < x_2 < 3b$ and $-3b < x_2 < -b$

$$\left| \frac{x_2+b}{x_2-b} \right| = \frac{x_2+b}{x_2-b} > 2 \qquad \left| \frac{x_2-b}{x_2+b} \right| = \frac{b-x_2}{-b-x_2} > 2$$

$\Rightarrow$ for $(x_1,x_2) \in R'$, we have $|S_H(\mathbb{1}_R)(x_1,x_2)| \geq \frac{\log 2}{2\pi} \geq \frac{1}{10}$. $\square$

We can now glue all pieces (exercises 7.3, 8.3, Prop 4.17, Lemma 4.16) together to prove

~~Theorem 5~~

Theorem 4.18 (Fefferman – Unboundedness of the ball multiplier in L^p, $p\neq 2$)

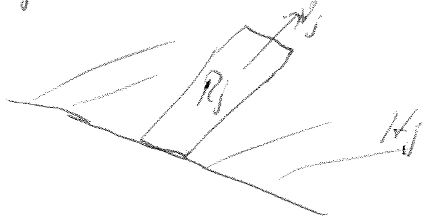
Let $d\geq 2$, then $\mathbb{1}_{B_0(1)}(\xi)$ is not an $L^p(\mathbb{R}^d)$ multiplier when $p\in(1,\infty)\setminus\{2\}$

Proof: By de Leeuw (ex 7.3), it suffices to consider $d=2$. By duality, wlog $p>2$.

Now assume $\mathbb{1}_{B_0(1)}(\xi)$ was $L^p(\mathbb{R}^2)$-bdd with operator norm $B_p<\infty$.

Suppose $\delta>0$ is given, then let E and R_j be as in Lemma 4.16.

Let $f_j=\mathbb{1}_{R_j}$; $v_j\in S^2$ be the direction of R_j and H_j the corresponding half-plane.



By Proposition 4.17, we obtain

$$\int_E \sum_j |S_{H_j}(f_j)|^2\,dx = \sum_j\int_E |(S_{H_j}f_j)(x)|^2\,dx \geq \sum_j\int_E \frac{1}{10^2}\mathbb{1}_{R_j'}(x)\,dx$$

$$= \frac{1}{100}\sum_j |E\cap R_j'| \underset{\text{Lemma 4.16 (iv)}}{\geq} \frac{1}{1200}\sum_j |R_j| \qquad (*)$$

$\Rightarrow$ By Hölder on the other hand, (with $(p/2)$ and $(p/2)'=\frac{p}{p-2}$)

$$\int_E \sum_j |S_{H_j}f_j|^2\,dx \leq |E|^{\frac{p-2}{p}} \underbrace{\Big\|\Big(\sum_j |S_{H_j}f_j|^2\Big)\Big\|_{p/2}}_{=\|(\sum|S_{H_j}f_j|^2)^{1/2}\|_{p/2}^2}$$

$$\underset{\text{Meyer}}{\leq} B_p^2\,|E|^{\frac{p-2}{p}}\,\Big\|\Big(\sum |f_j|^2\Big)^{1/2}\Big\|_p^2$$

$$\underset{R_j\text{'s disjoint}}{=} B_p^2\,|E|^{\frac{p-2}{p}}\Big(\sum|R_j|\Big)^{2/p}$$

$$\underset{\substack{|E|\leq\delta\sum|R_j|\\ \text{Lemma 4.16}}}{\leq} B_p^2\,\delta^{\frac{p-2}{p}}\sum|R_j|$$ which contradicts $(*)$ for δ sufficiently small.

$$\Big(\frac{1}{1200}\leq B_p^2\,\delta^{\frac{p-2}{p}}\Big)$$

$\boxtimes$

4.6 Littlewood-Paley theory

→ estimating kernels of Fourier multipliers and finding sufficient conditions for their L^p-boundedness

here we focus on Fourier multipliers which correspond to symbols of order 0

Our precise assumptions are as follows. (preliminary thm in the following..)

Let $m \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be a bounded function (in Fourier space) such that

$$|\partial_\xi^\alpha m(\xi)| \leq A_\alpha |\xi|^{-\alpha} \quad \text{for all } \alpha \qquad (a)$$

or, alternatively that $m \in C^\ell(\mathbb{R}^d \setminus \{0\})$ and satisfies

$$|\partial_\xi^\alpha m(\xi)| \leq A_\alpha |\xi|^{-\alpha}, \quad 0 \leq |\alpha| \leq \ell \quad \text{where } \ell = \lceil d/2 \rceil \qquad (b)$$

Theorem 4.18 (Mikhlin) (strictly speaking K is the distribution with ...)

a) Under assumption a), we have that the associated integral kernel K agrees with a function $K(x)$ away from $\vec{0}$ that is C^∞ there and satisfies

$$|\partial^\alpha K(x)| \leq A_\alpha' |x|^{-d-\alpha}, \quad \alpha \in \mathbb{N}^d$$

b) Under assumption b), K agrees with a function $K(x)$ away from $\vec{0}$, that is locally integrable there and satisfies

$$\int_{|x| > 2|y|} |K(x-y) - K(x)| \, dx \leq A, \quad y \neq 0.$$

→ ~~[illegible]~~ by theory of last notes, we get an (trivially) L^p-bdd transl. invariant operator

<u>Proof</u> a) We decompose our multiplier: let $\eta \in C_c^\infty(\widehat{\mathbb{R}^d})$ ↑Fourier space s.t. $\eta(\xi) = \begin{cases} 1 & |\xi| \leq 1 \\ 0 & |\xi| > 2 \end{cases}$

and $\delta(\xi) = \eta(\xi) - \eta(2\xi) = \begin{cases} \neq 0 & |\xi| \in [\frac{1}{2}, 2] \\ 0 & |\xi| \in [0, \frac{1}{2}] \cup [2, \infty) \end{cases}$

→ dyadic partition of unity $1 = \sum_{j=-\infty}^{\infty} \underbrace{\delta(2^{-j}\xi)}_{\text{supported in frequency shell } |\xi| \in [2^{j-1}, 2^{j+1}]} \quad \left(= \eta(\xi) + \sum_{j=1}^{\infty} \delta(2^{-j}\xi) \right)$

(in fact, $\eta(\xi) + \sum_{j=1}^{\ell} \delta(2^{-j}\xi) = \eta(2^{-\ell}\xi) \xrightarrow{\ell \to \infty} 1$,

resp. $\sum_{j=\ell'}^{\ell} \delta(2^{-j}\xi) = \eta(2^{-\ell}\xi) - \eta(2^{-\ell'+1}\xi) \xrightarrow[\ell' \to -\infty]{\ell \to \infty} 1$)

→ dyadic decomposition of m

$$m(\xi) = \sum_{j=-\infty}^{\infty} m(\xi)\,\delta(2^{-j}\xi) \equiv \sum_{j=-\infty}^{\infty} m_j(\xi)$$

→ let $K_j(x) := \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} m_j(\xi)\, d\xi$

Since $\sum_j m_j(\xi) \to m(\xi)$ for $\xi \neq 0$ (pointwise boundedly), it follows that $\sum K_j \longrightarrow K$ at least in the sense of distributions

⇒ suffices to estimate $\sum_j |\partial_x^\alpha K_j(x)|$ ($\geq |\partial_x^\alpha \sum K_j|$) for $x \neq 0$.

$$|\partial_x^\alpha K_j(x)| = |\partial_x^\alpha \int e^{2\pi i x\cdot\xi} m(\xi)\,\delta(2^{-j}\xi)\,d\xi|$$

$$\simeq_\alpha |\int e^{2\pi i x\cdot\xi}\, \xi^\alpha m(\xi)\,\delta(2^{-j}\xi)\,d\xi| \quad / \cdot \frac{|x|^M}{|x|^M} \quad (M \geq 0)$$

$$\lesssim_\alpha |x|^{-M} |\int \partial_\xi^M e^{2\pi i x\cdot\xi}\, \xi^\alpha m(\xi)\,\delta(2^{-j}\xi)\,d\xi|$$

$$\lesssim_{M,\alpha} |x|^{-M} \cdot 2^{+j(|\alpha|-M+d)}$$

⇒ for $M=0$ we obtain $\sum_{2^j < \frac{1}{|x|}} |\partial_x^\alpha K_j| \lesssim_\alpha \sum_{2^j < \frac{1}{|x|}} 2^{j(|\alpha|+d)} \sim |x|^{-d-\alpha}$

and for $M > d+\alpha$, $\sum_{2^j > \frac{1}{|x|}} |\partial_x^\alpha K_j| \lesssim_{M,\alpha} |x|^{-M} \sum_{2^j > |x|} 2^{j(|\alpha|+d-M)} \lesssim |x|^{-d-\alpha}$

thereby establisching a) (effectively only needs ~~$m \in C^{d+\alpha}$~~ $m \in C^M$, $M > d+\alpha$)

b) We use Plancherel instead of the crude estimates for the Fourier transform of m (m_j) → doesn't require so much regularity on m

First, one notes that

$$\int |(-2\pi i x)^\gamma K_j(x)|^2\, dx = \int \underbrace{|\partial_\xi^\gamma m_j(\xi)|^2}_{\lesssim |\xi|^{-M}}\, d\xi, \quad |\gamma| = M \text{ by Plancherel,}$$

$\int_{|\xi| \sim 2^j} |\xi|^{-2M} d\xi$

which implies

$$\int | |x|^M K_j(x)|^2\, dx \lesssim 2^{j(d-2M)}, \quad 0 \leq M \leq l \text{ by assumption b)}$$

$$\rightarrow \int_{|x| \leq a} |K_j(x)| \overset{CSU}{\leq} \underbrace{\sqrt{\int |K_j(x)|^2}}_{\lesssim 2^{jd/2}} \sqrt{\int_{|x|<a} dx} \lesssim 2^{jd/2} \cdot a^{d/2}, \quad a > 0.$$

Similarly, (*) $\int_{|x|>a} |K_j(x)|\,dx \leq \sqrt{\int (|K_j(x)|\,|x|^M)^2}\ \sqrt{\int_{|x|>a} |x|^{-2M}} \lesssim 2^{j(d/2-\ell)} \cdot a^{\frac{d}{2}-\ell}$, $a>0$,
$\ell = M$
= smallest integer bigger than $d/2$

$\Rightarrow$ Combining the last two estimates with $a = 2^{-j}$ (which one actually chooses at the end of the argument after an optimization over a), we obtain $\int_{\mathbb{R}^d} |K_j(x)| \leq A$, $j \in \mathbb{Z}$

Analogously, one proves that $\int_{\mathbb{R}^d} |\partial_x^\alpha K_j(x)| \leq A_\alpha\, 2^{\alpha j}$ for all j, α

For $|\alpha| = 1$ this yields in particular (by mean-value thm)

$$\int_{\mathbb{R}^d} |K_j(x+h) - K_j(x)|\,dx \leq A\,|h| \cdot 2^j \qquad (**)$$

Now, $\int_{|x|>2|y|} |K(x-y) - K(x)|\,dx \leq \sum_j \int |K_j(x-y) - K_j(x)| = \underbrace{\Sigma_1}_{\text{taken over } 2^j < |y|^{-1}} + \underbrace{\Sigma_2}_{\text{taken over } 2^j > |y|^{-1}}$

By (**), we estimate $\Sigma_1 \leq \sum_{2^j < \frac{1}{|y|}} \int |K_j(x-y) - K_j(x)|\,dx \leq |y| \sum_{2^j < |y|^{-1}} 2^j \sim 1$

Using (*), we estimate $\Sigma_2 \overset{\Delta\text{-ineq}}{\leq} \sum_{2^j > \frac{1}{|y|}} 2 \int_{|x|>|y|} |K_j(x)| \leq \sum_{2^j > \frac{1}{|y|}} 2^{j(\frac{d}{2}-\ell)}\, |y|^{d/2-\ell} \sim 1$ $\boxed{/}$

$|x+y| > 2|y| \Rightarrow |x| > |y|$

We now consider the converse question (once more..), namely, given a distribution kernel K which equals a function away from $\vec{0}$ which satisfies $|\partial_x^\alpha K(x)| \lesssim |x|^{-d-\alpha}$, what additional assumptions on K would suffice to guarantee that $\hat{K}$ is bounded, so that $f \mapsto K*f$ is an L^2-bdd operator?

$\to$ cancellation conditions of course (as before). Earlier, we assumed $\int_{R_1<|x|<R_2} K(x) = 0$, but this can be "upgraded".

Our characterization uses the notion of a "standard" or "normalized" bump function. Suppose we fix a large integer N (exact value not important, but any $N > d/2$ will do). Then $\phi \in C^\infty(\mathbb{R}^d)$ is said to be a <u>normalized bump fct</u> if $\operatorname{supp}\phi \subseteq B_0(1)_x$ and $|\partial_x^\alpha \phi| \le 1$ for all $0 \le |\alpha| \le N$.

Given ϕ, we denote by ϕ_R the scaled version $\phi_R(x) = \phi(x/R)$. The new cancellation condition will be expressed by testing K against ϕ_R. We have (testing in the sense of $\langle\cdot,\cdot\rangle_{\mathcal{S},\mathcal{S}'}$)

<u>Proposition 4.19</u> (Sufficient and necessary criterion for L^2-bddness)

Suppose K is a distributional kernel that away from the origin equals a function with $|\partial_x^\alpha K(x)| \le A_1 |x|^{-d-\alpha}$ for $|\alpha| \le 1$.

Then $\hat{K}$ is a bounded function if and only if there is $A>0$ s.t. $|K(\phi_R)| \le A$ for all normalized bump functions ϕ and all $R>0$.

<u>Proof</u> "$\Leftarrow$" Suppose $\hat{K}$ is bounded. Then (by $\widehat{\phi_R}(\xi) = R^d \hat{\phi}(R\xi)$),

$$K(\phi_R) = \int \hat{K}(\xi)\hat{\phi}(-R\xi) R^d \, d\xi = \int \hat{K}(-\xi/R)\hat{\phi}(\xi)\, d\xi \overset{(*)}{\le} \|\hat{K}\|_\infty \|\hat{\phi}\|_1 \lesssim A$$

$$\left(\text{since } \int |\hat{\phi}|\, d\xi \le \sqrt{\int |\hat{\phi}|^2 \langle\xi\rangle^{2N} d\xi} \underset{\text{Plancherel}}{\lesssim} \|\langle\partial_x\rangle^N \phi\|_2 \lesssim \sqrt{\int (|\phi| + |\partial_x^N \phi|)^2 dx} \lesssim 1\right)$$

"$\Rightarrow$" To prove $\|\hat{K}\|_\infty \lesssim 1$, we decompose $K = K_0 + K_\infty$ where $K_0 = K(x)\eta(x)$ with $\eta \in C_c^\infty(\mathbb{R}^d)$, $\operatorname{supp}\eta \subseteq B_0(1)$, $\eta = 1$ near the origin.

$\to$ $\hat{K} = \widehat{K_0} + \widehat{K_\infty}$ where both $\widehat{K_0}, \widehat{K_\infty} \in L^1_{loc}$ $\Rightarrow$ $\hat{K}$ is a function

bc $\widehat{K_0}$ is the FT of a compactly supported distribution (see also below the estimate on $|\widehat{K_0}(\xi)|$)

bc $\widehat{K_\infty}$ is the FT of the L^2-fct $\underbrace{(1-\eta)\cdot K}_{\le \langle\xi\rangle^{-d} \in L^2}$

Next, we claim that it suffices to prove

$(*)$ $|\hat{K}(\xi)| \le B$, $1 \le |\xi| \le 2$ where $B = B(A_1, A)$ (A, A_1 in assumptions)

Indeed, if $(*)$ holds, we can replace the distribution K with $K^{(\epsilon)}$ for any $\epsilon>0$ where $K^{(\epsilon)}\phi \equiv K(\phi_{1/\epsilon})$, i.e., the function $K_\epsilon(x)$ associated with K_ϵ satisfies

$K_\epsilon(x) = \epsilon^{-d} K(x/\epsilon) \Rightarrow$ if K satisfies $\begin{array}{l} |\partial_x^\alpha K(x)| \le A|x|^{-d-\alpha} \\ |K(\phi_R)| \le A \end{array}$ then K_ϵ satisfies

these bounds with the same ~~...~~ constants. So $(*)$ also implies also

$|\hat{K}(\epsilon\xi)| \le B = B(A_1, A)$ for $1 \le |\xi| \le 2$ and any $\epsilon > 0$; in other words, $\hat{K}$ is bounded.

So it remains to verify $(*)$. We again use the decomposition $K = K_0 + K_\infty$.

Since $2\pi i \xi_j \widehat{K_\infty}(\xi) = \int_{\mathbb{R}^d} \frac{\partial K_\infty}{\partial x_j} e^{-2\pi i x\cdot\xi}\,dx$ and $|\partial_{x_j} K_\infty| \le A' \langle x\rangle^{-d-1}$, we get

$|\widehat{K_\infty}(\xi)| \le B$ for $1 \le |\xi| \le 2$ (by multiplying and dividing the above by ξ_j)

Next, since K_0 is a compactly supported distribution, we have

$$\hat{K}_0(\xi) = K_0(e^{-2\pi i x\cdot\xi}) \equiv K(\psi), \quad \psi(x) = \eta(x) e^{-2\pi i x\cdot\xi}.$$

Observe that ψ satisfies the conditions $\left|\frac{\partial^\alpha \psi}{\partial x^\alpha}\right| \le 1$, $0 \le |\alpha| \le N$ for a normalized bump function, when $1 \le |\xi| \le 2$, possibly after multiplication with a suitable constant that is bounded away from zero.

$\Rightarrow$ the assumption $|K(\phi_R)| \le A$ implies, with $R=1$, $|\hat{K}_0(\xi)| \le B(A_1, A)$, $1 \le |\xi| \le 2$

$\Rightarrow$ $|\hat{K}(\xi)| \le 2B(A_1, A)$, $1 \le |\xi| \le 2$ what was claimed in $(*)$ $\square$

We now state a refinement of Thm 4.18.

Thm 4.20 (Hörmander multiplier thm)

Let $m \in L^\infty(\hat{\mathbb{R}}^d)$ and assume that, for some integer $s > d/2$,

$$\sum_{0 \le |\alpha| \le s} \sup_{\lambda > 0} \lambda^{-d} \| \xi \lambda \partial_\xi^{|\alpha|} \partial^\alpha \beta(\cdot/\lambda) m(\cdot) \|^2_{L^2(\mathbb{R}^d)} < \infty, \quad \beta \in C_c^\infty(\mathbb{R}^d)$$

Then, for any $1 < p < \infty$ we have that the associated operator

$T_m : f \mapsto (T_m f)(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} m(\xi) \hat{f}(\xi) d\xi$ satisfies

$$\| T_m f \|_{L^p(\mathbb{R}^d)} \le A_p \|f\|_{L^p(\mathbb{R}^d)} \quad \text{and} \quad \| T_m f \|_{L^{1,\infty}(\mathbb{R}^d)} \le A \|f\|_{L^1(\mathbb{R}^d)} .$$

- Note that multipliers satisfying $|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-\alpha}$ satisfy the above conditions.
- Examples of multipliers satisfying the "local smoothness" condition are imaginary powers (which are trivially L^2-bdd), i.e., $m(\lambda\xi) = \lambda^{i\tau} m(\xi)$ for $\tau \in \mathbb{R}$ and $m \in C_c^\infty$ (homogeneous of imaginary order)

(indeed, $\lambda^{|\alpha|} \partial_\xi^\alpha m(\lambda\xi) = \lambda^{i\tau} \partial_\xi^\alpha m(\xi) \overset{!!}{\le} |\xi|^{-\alpha}$

$\to$ taking $\lambda = |\xi|^{-1}$ we have $|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-\alpha}$ with $C_\alpha = \sup_{|\theta|=1} |\partial^\alpha m(\theta)|$

$\to$ explicit examples would be $m(\xi) = |\xi|^{i\tau}$ or $m(\xi_1, \xi_2, \xi_3) = \dfrac{\xi_1^2 + \xi_2^2}{\xi_1^2 + i(\xi_2^2 + \xi_3^2)}$

which is homogeneous of degree zero and smooth on $\mathbb{R}^3 \setminus \{0\}$

Exercise Check that $m_1(\xi) = \left(\dfrac{|\xi|^2}{1+|\xi|^2}\right)^z$ and $m_2(\xi) = \left(\dfrac{1}{1+|\xi|^2}\right)^z$ for $\operatorname{Re} z \ge 0$, $z \in \mathbb{C}$

are L^p Fourier multipliers $(1 < p < \infty)$

Another example are the Bochner-Riesz means $(1-|\xi|^2)_+^{\delta}$ ~~with~~ iff $\delta > s - 1/2$

(the above assumption can be relaxed to $\sup_{t>0} \| \omega(t\cdot) m(\cdot) \|_{H^s(\mathbb{R})} < \infty$

which is applicable for Bochner-Riesz)

(In fact this can be seen more directly by computing (exercise)

$$\int_{\mathbb{R}^d} (1-|\xi|^2)_+^\delta e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi \sim \int_{\mathbb{R}^d} \frac{\Sigma_\pm e^{\pm 2\pi i x \cdot \xi} + o_{|x-y| \to \infty}(1)}{1 + |x-y|^{(d+1)/2 + \delta}} f(y) dy$$

(at $x \to 0$, no problem since $m(\xi)$ compactly supported $\Rightarrow \hat{m}$ analytic)

and so one sees that the kernel is in L^1 for $\delta > \frac{d-1}{2}$, so Schur's test is applicable!)

Proof By Plancherel, $\|T_m f\|_2 = \|m \hat{f}\|_2 \le \|m\|_\infty \|f\|_2$, i.e., T is L^2-bdd.

Next, by ~~self-adjointness~~ duality, i.e., $\int \overline{g(x)}\,(T_m f)(x)\,dx = \int \overline{(T_{\bar m} g)(x)}\, f(x)\,dx$

and $\bar m$ satisfies the same assumptions as m, it suffices to prove the Thm for $p \in (1,2)$. Finally, by Marcinkiewicz interpolation, it suffices to prove the $L^1 \to L^{1,\infty}$-bound. As is customary, we perform a Calderón-Zygmund decomposition at level α say, i.e.,

$$f = g + \sum b_k, \quad \|g\|_\infty \lesssim \alpha, \quad b_k \text{ supported on non-overlapping } Q_k$$

$$\int b_k = 0, \qquad |\Omega| = |\bigcup Q_k| \le \frac{\|f\|_1}{\alpha}$$

By the usual arguments, we only need to consider

$|\{x \in (\Omega^*)^c : |(T_m b)(x)| > \alpha\}|$ and so we're done as soon as we show $\int_{(Q_k^*)^c} dx\, |(T_m b_k)(x)| \lesssim \|b_k\|_{L^1}$ (by Chebyshev)

$\Omega^* = \bigcup_k Q_k^*$ ← $2\sqrt{d}$-dilates

To this end we decompose $K = \check{m}$ dyadically. Let $\psi \in C_c^\infty(B_0(2))$ with $\psi(\xi) = 1$, $|\xi| \le 1$ and define $\beta(\xi) = \psi(\xi) - \psi(2\xi)$ being a bump located at $|\xi| = 1$. Then $1 = \sum_{j=-\infty}^{\infty} \beta(2^{-j}\xi)$ and

$$m(\xi) = \sum_{j=-\infty}^{\infty} m(\xi)\beta(2^{-j}\xi) \equiv \sum_{j=-\infty}^{\infty} m_{2^j}(2^{-j}\xi), \quad m_\lambda(\xi) \equiv \beta(\xi)\, m(\lambda \xi)$$
(here $\lambda = 2^j$)

Let further K_λ be defined by $\hat{K}_\lambda = m_\lambda \in \mathcal{S}$.

$\Rightarrow$ By $m_\lambda(\xi/\lambda) = (\lambda^d K_\lambda(\lambda\,\cdot))^\wedge$, we have

$$K(x) = \sum_{j=-\infty}^{\infty} 2^{jd} K_{2^j}(2^j x)$$ where convergence of the sum is understood in $\mathcal{S}'$-sense.

Now scaling the assumption $\sum_{0 \le |\alpha| \le s} \sup_{\lambda > 0} \lambda^{-d} \| \lambda^{|\alpha|} D^\alpha \beta(\cdot/\lambda)\, m(\cdot)\|_2^2 < \infty$ ($\xi \mapsto \lambda\xi$) means (with the above terminology $m_\lambda(\xi) = \beta(\xi)\, m(\lambda\xi)$) that

$$\sup_{\lambda > 0} \int |(1-\Delta_\xi)^{s/2} m_\lambda(\xi)|^2\, d\xi < \infty$$

$$\overset{\text{Plancherel}}{\Longleftrightarrow} \sup_{\lambda > 0} \int |(1+|x|^2)^{s/2} K_\lambda(x)|^2\, dx < \infty$$

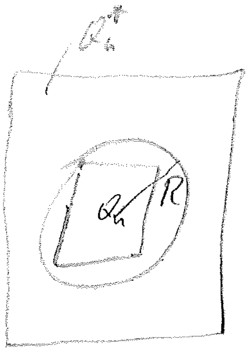
By CSU, $\int_{\{x:\ \max|x_j|>R\}} |K_\lambda(x)|\,dx \lesssim \sqrt{\int_{|x|>R} \langle x\rangle^{-2s}\,dx} \sim R^{d/2-s}$ for any $R>0$
(and $s>\frac{d}{2}$)

Moreover, since $\xi_j m_\lambda(\xi)$ satisfies the same estimates as $m_\lambda(\xi)$, we obtain analogously $\int |\nabla K_\lambda(x)|\,dx < \infty \Rightarrow$ by mean-value thm $\int dx\,|K_\lambda(x+y)-K_\lambda(x)| \lesssim |y|$

~~Now, after possibly making a possible translation, we can choose, for fixed k, R s.t.~~ ~~$Q_k = \{x : \max|x_j| \le R\}$~~

$T_m = \sum_j T_{m_j}$ ← distinguish between small and large j wrt diam Q_k.

Recall our desired estimate $\int_{(Q_k^*)^c} dx\,|(T_m b_k)(x)| \lesssim \|b_k\|_{L^1}$ which would follow from



$\sup_{y\in Q_k} \int_{(Q_k^*)^c} dx\,|K(x-y)| \lesssim 1$ by Young, for instance

$$\lesssim \sup_{y\in Q_k} \int_{(Q_k^*)^c} dx \sum_j 2^{jd}|K_{2^j}(2^j(x-y))|$$

$$\lesssim \sup_{y\in\mathbb{R}^d}\ \sup_{R>0} \int_{(B_y(R))^c} dx \sum_j 2^{jd}|K_{2^j}(2^j(x-y))|$$

$$= \sup_{y\in\mathbb{R}^d}\ \sup_{R>0} \int_{(B_y(2^jR))^c} dx\, \sum_j |K_{2^j}(x-y)|$$

$$\lesssim \sup_{y\in\mathbb{R}^d}\ \sup_{R>0} \sum_j (2^jR)^{d/2-s}$$ which would be finite if the j-summation started at $1/\log_2 R$ ($\Leftarrow 2^jR \ge 1$)

large j (compared to log diam Q_k)

small j (compared to log diam Q_k)

So what happens for $2^jR \le 1$?

→ Using that $\int b_k = 0$, we even have (recalling $K(x) = \sum_{j\in\mathbb{Z}} 2^{jd}K_{2^j}(2^jx)$)

newly inserted

$$\|T_{m_j} b_k\|_{L^1} = \int_{\mathbb{R}^d} dx \left|\int_{\mathbb{R}^d} dy\, 2^{jd}\left(K_{2^j}(2^j(x-y)) - K_{2^j}(2^jx)\right) b_k(y)\right| \underset{x\mapsto \frac{x}{2^j}}{\lesssim} \int dy\, |b_k(y)| \cdot 2^j \underbrace{|y|}_{\le R} \le 2^jR$$

and the right side is summable in j for $2^jR \le 1$ $\boxed{=}$

(Basic idea: decompose multiplier dyadically where the distinction between "high" and "low" frequencies highly depends on the scale of supp b_k (~ uncertainty principle)

TODO

One of the most powerful consequences of multiplier thms in applications (e.g., non-linear PDE) are square function estimates.

As before, let $\beta \in C_c^\infty(\mathbb{R}^d)$, $\operatorname{supp} \beta \subseteq \{\xi : \frac{1}{2} < |\xi| < 2\}$, $1 = \sum_{j \in \mathbb{Z}} \beta(2^{-j}\xi)$. Then we define the Littlewood-Paley projection P_j by

$$(P_j f)^\wedge(\xi) = \beta(\xi/2^j)\, \hat{f}(\xi)$$

($\leftarrow$ smooth; one can also think of rough cutoffs à la Hilbert transform and Marcinkiewicz-Zygmund)

Let $(Sf)(x) = \left(\sum_{j \in \mathbb{Z}} |(P_j f)(x)|^2\right)^{1/2}$. Then, we have

Thm 4.21 (Littlewood-Paley inequalities)

Let $1<p<\infty$. Then there's a constant A_p such that

$$A_p^{-1} \|Sf\|_p \underset{\substack{\uparrow \\ \text{square fct estimate}}}{\leq} \|f\|_p \overset{\substack{\text{reverse square fct estimate} \\ \swarrow}}{\leq} A_p \|Sf\|_p$$

Proof We show how the first inequality implies the second one (by duality).

Since $\beta(\xi/2^j)\,\beta(\xi/2^{j-\ell}) = 0$ for all $|\ell|>1$, we have, by Plancherel

$$\langle g, f \rangle = \int_{\mathbb{R}^d} \bar{g} f \, dx = \sum_{\substack{j,\ell \\ |j-\ell| \leq 1}} \int_{\mathbb{R}^d} \overline{(P_\ell g)(x)}\, (P_j f)(x)\, dx$$

$$\underset{\text{CSU in } j,\ell}{\overset{|\cdot|}{\leq}} \int dx \underbrace{\sqrt{\sum_{j,\ell} |(P_\ell g)(x)|^2 \mathbb{1}_{|j-\ell| \leq 1}}}_{= \sqrt{2}\,(Sg)(x)} \underbrace{\sqrt{\sum_{j,\ell} |(P_j f)(x)|^2 \mathbb{1}_{|j-\ell| \leq 1}}}_{\sqrt{2}\,(Sf)(x)}$$

$$\overset{\text{Hölder}}{\lesssim} \|Sg\|_{p'} \, \|Sf\|_p \underset{\substack{\uparrow \\ \text{square fct est.} \\ \text{for } Sg}}{\leq} A_{p'} \|g\|_{p'} \, \|Sf\|_p$$

$\rightarrow$ by duality of the L^p spaces, i.e., taking the $\sup_{\|g\|_{p'}=1}$, we obtain

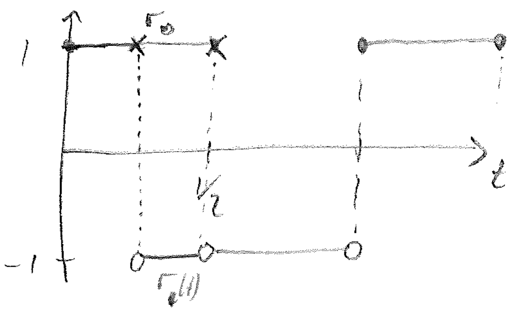
$\|f\|_p \leq A_{p'} \|Sf\|_p$, i.e., the reverse square fct. estimate.

To finish the proof, we're left to show $\|Sf\|_p \lesssim \|f\|_p$.

We'll utilize a randomization argument similarly as in the proof of Khinchin's inequality, namely Rademacher functions ($\sim$ Rademacher signs)

$r_0(t) := \begin{cases} 1 & 0 \le t \le 1/2 \\ -1 & 1/2 < t < 1 \end{cases}$ and extend periodically, i.e., $r_0(t) = r_0(t-1)$

$r_j(t) := r_0(2^j t)$



$\to$ the $r_j(t)$ are obviously orthonormal on $[0,1]$ (by scaling it suffices to consider
$$\int_0^1 r_0(t) r_0(2^j t)\,dt = \int_0^{1/2} r_0(2^j t) - \int_{1/2}^1 r_0(2^j t)\,dt = 2^{-j}\left[\int_0^{2^{-1-j}} r_0(t) - \int_{2^{-1-j}}^{2^{-j}} r_0(t)\,dt\right] = 0$$

Moreover, if $F(t) = \sum_j a_j r_j(t) \in L^2([0,1])$, then automatically $F \in L^p([0,1])$ for all $0 < p < \infty$ by Khinchine's inequality (p. 4-5 in notes 1) which can be proven completely analogously and states
$$A_p^{-1} \|F\|_{L^p([0,1])} \le \|F\|_{L^2([0,1])} = \sqrt{\sum |a_j|^2} \le A_p \|F\|_{L^p([0,1])}$$
(for an explicit proof using Rademacher fcts, see, e.g., Stein Appendix D)

We'll now prove the square fct estimate using Khinchine. We define a transl. op. T_t ($t \in [0,1]$) by
$$(T_t f)^\wedge(\xi) \equiv m_t(\xi) \hat f(\xi) = \overbrace{\sum_{j=0}^{\infty} r_j(t) \beta(\xi/2^j) \hat f(\xi)}^{\text{falls under \sout{the} scope of Hörmander since } |D_\xi^\gamma m_t(\xi)| \lesssim_\gamma |\xi|^{-\gamma}}$$

Then, by Khinchine, $\sqrt{\sum_{j\ge 0} |\beta(\xi/2^j) \hat f(\xi)|^2} \lesssim \left(\int_0^1 |(T_t f)^\wedge(\xi)|^p\,dt\right)^{1/p}$ $\quad / \ |\cdot|^p, \int d\xi$, Plancherel, $\sqrt{\cdot}$

$\Rightarrow$ ~~NB~~ and analogously (taking FT) $\sqrt{\sum_{j\ge 0} |(P_j f)(x)|^2} \lesssim \left(\int_0^1 |(T_t f)(x)|^p\,dt\right)^{1/p}$

Now $|\cdot|^p$, $\int dx$ gives $\|Sf\|_p^p \lesssim \int_0^1 dt \int_{\mathbb{R}^d} dx\, |(T_t f)(x)|^p \underset{\text{Hörmander}}{\lesssim} \|f\|_p^p$ and analogously for the $j < 0$. ∎

We will now generalize these considerations to the setting of non-negative, i.e., self-adjoint operators. Our guiding example will be $-\Delta + V$ or variations thereof. We begin with a fundamental result of Hebisch where $V \geq 0$. For generalizations, see., e.g., Blunck, Blunck-Kunstmann, Chen-Ouhabaz-Sikora-Yan (i.e., V with negative part). The case $|p|^\alpha + V$ is considerably harder and only partial results are available so far. 30

<u>Theorem 4.22</u> (Spectral multiplier thm for $-\Delta+V$ in $L^2(\mathbb{R}^d)$)

Suppose $0 \leq V \in L^1_{loc}$ and that $A \equiv -\Delta + V$ is defined in the sense of quadratic forms. Let $\epsilon > 0$ and fix $0 \neq \varphi \in C_c^\infty(\mathbb{R}_+)$. Let $F \in H_{loc}^{\frac{d+1}{2}+\epsilon}(\mathbb{R}_+)$, i.e., Assume that $\|\varphi(\cdot) F(t\,\cdot)\|_{H^{\frac{d+1}{2}+\epsilon}(\mathbb{R})} \leq 1$ for all t. Then $F(A)$ defined via the functional calculus, i.e., $F(A) = \int_{\sigma(-\Delta+V)} F(\lambda)\, dE_{-\Delta+V}(\lambda)$ has weak-type (1,1) and is $L^p(\mathbb{R}^d)$-bdd for all $1 < p < \infty$.

<u>Preliminaries</u> 1) $e^{-tA}(x,y) \leq e^{t\Delta}(x,y)$ by Trotter's formula or Duhamel

$$e^{-t(-\Delta+V)} = s\text{-}\lim_{n\to\infty} \left(e^{-t(-\Delta)/n} e^{-tV/n}\right)^n$$

as a kernel, $\int dy_1 dx_2\, e^{+t\Delta/3}(x_1,y_1)\, e^{-tV/3}(y_1)\, e^{+t\Delta/3}(y_1,x_2)$
$e^{-tV/3}(x_2)\, \underbrace{e^{+t\Delta/3}(x_2,y_2)}_{>0}\, \underbrace{e^{-tV/3}(y_2)}_{\in[0,1]}$

$$\leq e^{t\Delta}(x_1, y_2)$$

Duhamel $e^{\Delta} - e^{-(-\Delta+V)} = \underbrace{\int_0^1 e^{+(1-s)\Delta} V e^{-s(-\Delta+V)}\, ds}_{\geq 0}$ (<u>exercise</u>)

So $e^{-(-\Delta+V)} = e^{\Delta} - \int \ldots \leq e^{\Delta}$

Since $e^{t\Delta} \sim \frac{e^{-|x|^2/t}}{t^{d/2}}$

2) $\int_{\mathbb{R}^d} |e^{-tA}(x,y)|^2\, dx \sim t^{-d/2}$ ($L^2 \to L^\infty$-bdd $\Leftrightarrow$ ultracontractive)

3) $\int e^{-tA}(x,y)\, e^{s|x-y|} \underset{x \mapsto \sqrt{t}}{\lesssim} e^{cs^2 t}$ for some c

(basically by completing the square)

(1)

Hebisch - A multiplier theorem for Schrödinger ops.

CENTRE DE RECHERCHES MATHÉMATIQUES

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Let $q > \frac{d}{2}$, $\varphi \in C_c^\infty(\mathbb{R}_+)$, $\varphi \neq 0$, $F(A)f = \int F(\lambda)\, dE(\lambda) f$ (s.a. op. with spectral measure $E(\lambda)$), $F_t(x) = F(tx)$ s.t. $\sup_{t>0} \|\varphi F_t\|_{H^q} < \infty$

I.p. $A = -\Delta + V$ in $L^2(\mathbb{R}^d)$ with $V \geq 0$ $\Rightarrow e^{-tA}(x,y) \leq e^{t\Delta}$ by Trotter's formula or the maximum principle or Duhamel

Since $e^{t\Delta}(x,y) = p_t(x-y) = (4\pi t)^{-d/2} e^{-|x-y|^2/4t}$, we have

(5) $\int e^{-tA}(x,y) e^{s|x-y|} dx \lesssim e^{cs^2 t}$

(6) $\int |e^{-tA}(x,y)|^2 dy \lesssim t^{-d/2}$ $\forall y \in \mathbb{R}^d$ ($L^2 \to L^\infty$-bdd, i.e., by ultracontractivity)

($\Rightarrow$) (7) $\sup_{x,y} |e^{-tA}(x,y)| \leq t^{-d/2}$ (by duality $\Rightarrow$ ultracontractive)

Define $\|K\|_a := \max\{ \sup_x \int |K(x,y)| (1+|x-y|)^a dy, \sup_y \int |K(x,y)| (1+|x-y|)^a dx \}$

Thm Let $\varepsilon > 0$, $0 \neq \varphi \in C_c^\infty(\mathbb{R}_+)$ s.t. $\|\varphi F_t\|_{H^{\frac{d+1}{2}+\varepsilon}} \leq c$ $\forall t$ $\Rightarrow F(A)$ is of weak type (1,1) and hence L^p-bdd $\forall 1<p<\infty$ (by Marcinkiewicz & duality)

Pf Do a Calderón-Zygmund decomposition of $f \in L^1$ at level λ, i.e., $\exists$ fcts b_i and g and disjoint cubes Q_i s.t.

$$f = g + \sum b_i, \quad \operatorname{supp} b_i \subseteq Q_i, \quad \int |b_i| \lesssim \lambda |Q_i|$$
$$|g(x)| \lesssim \lambda, \quad \sum |Q_i| \lesssim \|f\|_1 / \lambda$$

Introduce $\varphi \in C_c^\infty(\mathbb{R})$, $\operatorname{supp} \varphi \subset [\frac{1}{4}, 2]$, $\sum \varphi(2^{2k} x) = 1$ $\forall x > 0$

$\psi \in C_c^\infty(\mathbb{R})$, $\operatorname{supp} \psi \subseteq [-1,1]$ with $\psi(x) = 1$ for $x \in [0, \frac{1}{2}]$

and define $F_k(\lambda) := \varphi(2^{2k}\lambda) F(\lambda)$ and $\psi_k(\lambda) := \psi(2^{2k}\lambda)$

If $j < k \Rightarrow \psi_k F_j = 0 \Rightarrow \psi_k(A) F_j(A) = 0$ ($\lambda \leq 2^{-2k}$ & $\lambda \geq \frac{1}{4} 2^{-2j} = 2^{-2j-2}$)

If $j > k \Rightarrow \psi_k(A) F_j(A) = F_j(A)$

Thus, $F(A)f = \sum_{j,i} F_j(A) b_i + F(A) g$ $\qquad k_i = [\log_2(\operatorname{diam} Q_i)]$ $\qquad \Psi_k F_j = \begin{cases} 0 & j<k \\ F_j & j\ge k \end{cases}$

$$= \sum_i \left(\sum_{j\le k_i}' F_j(A) b_i + \sum_{j>k_i} F_j(A) b_i \right) + F(A) g$$

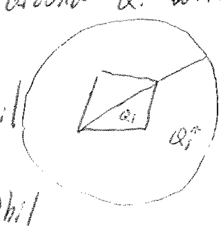
from sum $\sum_{j\le k_i} F_j \Psi_{k_i} = F_{k_i} \Psi_{k_i}$

$$= \sum_i \sum_{j\le k_i} F_j(A) b_i + \sum_{i,j} F_j(A) \Psi_{k_i}(A) b_i - \sum_i F_{k_i}(A) \Psi_{k_i}(A) b_i + F(A) g$$

$$= \sum_i \sum_{j\le k_i} F_j(A) b_i + F(A) \left(\sum_i \Psi_{k_i}(A) b_i + g \right) - \sum_i F_{k_i}(A) \Psi_{k_i}(A) b_i$$

Putting $S = \bigcup_i Q_i^*$, where Q_i^* is the concentric ball around Q_i with same center and radius $= 2\operatorname{diam} Q_i$,

we have $\cdot\ |\{x : |\sum_i \sum_{j\le k_i} F_j(A) b_i| > \frac{\lambda}{3}\}| \overset{\text{Chebyshev}}{\le} |S| + \frac{3}{\lambda} \int_{S^c} |\sum_i \sum_{j\le k_i} F_j(A) b_i|$



$$\lesssim \|f\|_1 / \lambda + \frac{3}{\lambda} \int_{S^c} |\sum_i \sum_{j\le k_i} F_j(A) b_i|$$

$\cdot\ |\{x : |F(A)(\sum_i \Psi_{k_i}(A) b_i + g)| > \frac{\lambda}{3}\}| \lesssim \lambda^{-2} \|\sum_i \Psi_{k_i}(A) b_i + g\|_2^2 \lesssim \lambda^{-2}\left(\|f\|_1 \cdot \lambda + \|\sum_i \Psi_{k_i}(A) b_i\|_2^2\right)$

$\cdot\ |\{x : |\sum F_{k_i}(A) \Psi_{k_i}(A) b_i| > \frac{\lambda}{3}\}| \lesssim \lambda^{-1} \|\sum_i F_{k_i}(A) \Psi_{k_i}(A) b_i\|_{L^1}$

We show in the following (A) $\int_{S^c} |\sum_i \sum_{j\le k_i} F_j(A) b_i|(x) dx \lesssim \sum_i \|b_i\|_{L^1} \lesssim \|f\|_1$

(B) $\|\sum_i \Psi_{k_i}(A) b_i\|_2^2 \lesssim \lambda \|f\|_{L^1}$

(C) $\|\sum_i F_{k_i}(A) \Psi_{k_i}(A) b_i\|_{L^1} \lesssim \sum_i \|b_i\|_{L^1} \lesssim \|f\|_{L^1}$

Define $\|K\|_{B(\phi)} := \max\{ \sup_x \int |K(x,y)|\, \phi(x,y)\, dy,\ \sup_y \int |K(x,y)|\, \phi(x,y)\, dx \}$ for a submultiplicative function ϕ (i.e. $\phi(x,y) \geq 1$, $\phi(x,z)\phi(z,y) \geq \phi(x,y)$)

examples: $\phi(x,y) = e^{a|x-y|}, (1+|x-y|)^a, e^{a|x-y|}(1+|x-y|)^b, \dots$

<u>Lemma 1</u> Assume that $K = K^*$, $\|K\|_{B(e^{b|x-y|})} \leq c_0$, and $\sup_x \int |K(x,y)|^2 dy \leq c_0$

$\Rightarrow$ for every $a \geq 0$ $\exists$ a constant $c = c(a,b,c_0)$ s.t. $\|e^{inK} K\|_a \leq c(1+|n|)^{a+d/2}$

<u>Proof</u> For $c<b$ and fixed a, define $\phi(x,y) = e^{c|x-y|}(1+|x-y|)^a \Rightarrow \|K\|_{B(\phi)} \leq c\|K\|_{B(e^{b|x-y|})} \leq cc_0$

Let $\ell = c^{-1}|n|\, \|K\|_{B(\phi)}$. Since e^{inK} is unitary,

$(\|e^{inK}K\|_b \leq \|e^{inK}\|_b \|K\|_b)$ $(1+n)^K$ if $\|K\|_a < \infty$ $(a \geq b)$

$$\int |(e^{inK}K)(x,y)|\,(1+|x-y|)^a\, dx = \int_{|x-y|<\ell} \dots + \int_{|x-y|>\ell} \dots$$

$$\leq (1+\ell)^a |B(y,\ell)|^{1/2} \|e^{inK}K\|_{L^2(B(y,\ell))} + e^{-c\ell} \|e^{inK}K(1+|\cdot - y|)^a e^{c|\cdot - y|}\|_{L^1}$$

$$\leq (1+\ell)^{a+d/2} \|e^{inK}K\|_{L^2} + e^{-c\ell}\|e^{inK}K\|_{B(\phi)}$$ (more precisely $\|K(\cdot,y)\|_{L^2}$)

$\ell \leq |n|c^{-1}cc_0$; $\|K\|_{L^2} < c_0$; submultiplicativity of ϕ

$$\leq (1+|n|)^{a+d/2} + \|K\|_{B(\phi)} \|e^{inK}\| e^{|n|\|K\|_{B(\phi)}} e^{-|n|\|K\|_{B(\phi)}} \lesssim (1+|n|)^{a+d/2}$$ ∎

<u>Lemma 2</u> If supp $F \subset [-1,1]$, $\epsilon > 0$, $a \geq 0$ $\Rightarrow \|F(A)\|_a \lesssim \|F\|_{H^{(d+1+\epsilon)/2+a}}$

<u>Proof</u> Let $K(\mu) = F(-\log \mu)\mu^{-1}$, then $\|K\|_{H^q} \lesssim \|F\|_{H^q}$ (e.g. if $q = \frac{d+1+\epsilon}{2} + a$) and supp $K \subseteq [e^{-1}, e]$

Decompose $K(\mu) = \sum_n \hat{K}(n) e^{in\mu} \Rightarrow F(A) = K(e^{-A})e^{-A} = \sum_n \hat{K}(n) e^{ine^{-A}} e^{-A}$

By Lemma 1 $(K = e^{-A})$, we have (and by the exp. bounds on e^{A}) $\|e^{ine^{-A}}e^{-A}\|_a \lesssim (1+|n|)^{a+d/2}$

$$\Rightarrow \|F(A)\| \lesssim \sum |\hat{K}(n)|(1+|n|)^{a+d/2} \leq \left(\sum |\hat{K}(n)|^2 (1+|n|)^{d+2a+1+\epsilon}\right)^{1/2} \left(\sum (1+|n|)^{-1-\epsilon}\right)^{\frac{1}{2}}$$

$$\lesssim \|K\|_{H^{a+(d+1+\epsilon)/2}} \lesssim \|F\|_{H^{a+(d+1+\epsilon)/2}}$$ ∎

Recall $F_k(\lambda) = \varphi(2^{2k}\lambda)F(\lambda)$; by dilation, we obtain for $a<\varepsilon$ $\int |F_k(A)(x,y)|(1+2^{-k}|x-y|)^a dx \leq c$

Pf. Let $\delta_t x := tx$, $\delta_t f := f\circ\delta_t$ i.e. $(\delta_t f)(x) = f(tx)$. Then $\cancel{\delta_{2^k} A \delta_{2^{-k}} =}$

$$\delta_{2^k} A \delta_{2^{-k}} = \delta_{2^k}(-\Delta)\delta_{2^{-k}} + \delta_{2^k} V \delta_{2^{-k}} = 2^{-2k}(-\Delta) + (V\circ\delta_{2^k}) \equiv 2^{-2k} A_k$$

$\delta_t V(x) f(2^{-k}x) = V(2^k\cdot)f(x)$

with $A_k = -\Delta + 2^{2k}(V\circ\delta_{2^k})$

$\Rightarrow F_k(A) = \cancel{\varphi(2^{2k}A)F(A) = \delta_{2^{-k}}(F_k\circ\delta_{2^{-2k}})(A)\delta_{2^k}}$ $\delta_{2^{-k}}(\underbrace{F_k\circ\delta_{2^{-2k}}}_{\tilde F_k})(\overbrace{A_k}^{\delta_{2^k}2^{2k}A})\delta_{2^k}$

$\overset{\text{Hebisch}}{=} \delta_{2^{-k}}\tilde F_k(A_k)\delta_{2^k}$ $\qquad \tilde F_k(A_k) = \delta_{2^k} F_k(A)\delta_{2^{-k}}$

$\Rightarrow$ apply $\|F(A)\|_a \lesssim \|F\|_{H^{a+(d+1+\varepsilon)/2}}$ with F replaced by $\tilde F_k$ and A by A_k, i.e., $\tilde F_k = F_k\circ\delta_{2^{-2k}}$

$\cancel{\|F_k(A)\|_a = \|\delta_{2^{-k}}\tilde F_k(A_k)\delta_{2^k}\|}$

$$\sup_x \int |F_k(A)(x,y)|\left(1+\frac{|x-y|}{2^k}\right)^a dy = \sup \delta_{2^{-k}} \underbrace{\int |\tilde F_k(A_k)(x,y)|(1+|x-y|)^a dy}_{\leq \|\tilde F_k\|_{H^{a+(d+1+\varepsilon)/2}}} \lesssim \|\tilde F_k\|_{H^{a+(d+1+\varepsilon)/2}} \approx \|\varphi F_{2^{-2k}}\|_H$$

$\forall x\in\mathbb{R}^d$ $\boxed{\checkmark}$

Frage: Träger von $\tilde F_k$ unabh. von k? (als Fkt von λ!)

(R_k ringt nicht in Abhängigkeit von, da nur die Fkt hinein eingeht und diese mit Maximumsprinzip abgeschätzt wird)

$\cancel{\tilde F_k(\lambda) = \varphi(\lambda)F(\lambda) \ \int \ldots (1+|\cdot|)^a}$

$\to$ ja! $\tilde F_k(\lambda) = \varphi(\lambda)F(2^{-2k}\lambda) \equiv \varphi(\lambda) F_{2^{-2k}}(\lambda)$

p.V. im Theorem ist $\sup_t \|\varphi F_t\|_{H^q} < \infty$!

$\boxed{\checkmark}$

$\varphi(A_j)F(2^{-2j}A)$

$$\tilde F_j(A_j) = \sum_n \hat h(n) e^{ine^{-2^{2j}A_j}} \ldots$$

$\cancel{= \sum_n \hat h(n) e^{ine^{-\delta_{2^j}A\delta_{2^{-j}}}} e^{-\delta_{2^j}A\delta_{2^{-j}}}}$

$2^{-2j}A_j = \delta_{2^j}A\delta_{2^{-j}}$

If $h\in L^1(\mathbb{R}^d)$ with $\operatorname{supp} h \subseteq \{x : |x|\leq 1\} = B_1$, then

$$\int_{|x|>2} |(F_k(A)h)(x)|dx \leq \|h\|_1 \sup_{y\in B_1} \int_{|x|>2} |F_k(A)(x,y)| \cdot 2^{ak}\cdot 2^{-ak} dx$$

$\cancel{\leq 2^{ka}\|h\|_1 \sup_y \int |F_k(A)(x,y)|(1+2^{-k}|x-y|)^a}$ $\quad$ b/c $|x-y|>1$ in step before

$$\leq 2^{ka}\|h\|_1 \sup_y \int_{|x-y|>1} |F_k(A)(x,y)|(1+2^{-k}|x-y|)^a$$

$$\leq 2^{ka}\|h\|_1 \sup_y \int |F_k(A)(x,y)|(1+2^{-k}|x-y|)^a \lesssim 2^{ak}\|h\|_1$$

2^j geht in Dilatation

(F_j): $\delta_{2^j}\tilde F_j(A_j)\delta_{2^j} \left(1+\frac{|x-y|}{2^j}\right)^a$

and thus $\sum_{k\leq 0} \int_{|x|>2} |(F_k(A)h)(x)|dx \lesssim \|h\|_1 \sum_{k\leq 0} 2^{ak} \lesssim \|h\|_1$

$$\Rightarrow \int_{(Q_i^*)^c} |(F_j(A)b_i)(x)|dx \leq \|b_i\|_1 \sup_{y\in Q_i} \int_{(Q_i^*)^c} |F_j(A)(x,y)| 2^{a(j-k_i)} 2^{-a(j-k_i)} \leq \|b_i\|_1 2^{a(j-k_i)} \sup_y \int_{|x-y|>2^{k_i}} |F_j(A)(x,y)| (\ldots)$$

$2^{k_i} \lesssim |x-y|$ $\to$ $\cancel{\text{scale } \ldots 2^{k_i}}$ $\cancel{\Rightarrow \text{Integral unit. } \ldots}$, $\sum_{j\leq k_i}$ $\Rightarrow$ (A)!

$(\frac{1}{2}+|x-z|+|z-y|)^2 \lesssim 2(\frac{1}{2}+|x-z|)^2 + (\frac{1}{2}+|z-y|)^2$

Lemma 3 $\forall m \geq 0\ \exists N, C>0$ s.t., if $F \in H^N$ with $\operatorname{supp} F \subset [-1,4]$, then
$|F(A)(x,y)| \leq C\|F\|_{H^N}(1+|x-y|)^{-m} \quad \forall x,y \in \mathbb{R}^d, A$

Pf Put $G(\lambda)=F(\lambda)e^{\lambda}$ and $N = d/2+m+1$, then $\|G\|_{H^N} \lesssim \|F\|_{H^N}$.

By Lemma 2, $\|G(A)\|_m \lesssim \|F\|_{H^N}$ (actually $\|G\|_{H^{m+(d+1+\varepsilon)/2}}$)
By the ~~pointwise bound~~ ultracontractivity $e^{-tA}(x,y) \lesssim t^{-d/2}$ $\forall x,y \in \mathbb{R}^d$, we have

$$(1+|x-y|)^m F(A)(x,y) = \left|\int G(A)(x,z)\, e^{-A}(z,y)\,(1+|x-y|)^m\, dz\right|$$

$$\lesssim \int |G(A)(x,z)|\,(1+|x-z|)^m\, e^{\Delta}(z-y)\,(1+|z-y|)^m\, dz$$

$$\leq \|G(A)\|_m \underbrace{\sup_z e^{-|z-y|^2}(1+|z-y|)^m}_{=\sup_z e^{-|z|^2}(1+|z|)^m \leq C} \qquad \blacksquare$$

$\Psi_k(\lambda) = \Psi(2^{2k}\lambda)$

Recall $\Psi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\Psi \subset [-1,1]$, $\Psi(x)=1$ for $x \in [0,\frac{1}{2}]$. Then, by dilation and Lemma 3 $(m>d)$,
~~$\|(\Psi_k F_k)(A)\|_{L^1\to L^1} \lesssim 1$~~ $\big($ $F_k(\lambda) = \varphi(2^{2k}\lambda)F(\lambda)$, $A_k = -\Delta + 2^{2k}V\circ\delta_{2^k}$, $\delta_{2^k} A \delta_{2^{-k}} = 2^{-2k}A$,
$\tilde{F}_k = F_k \circ \delta_{2^{-2k}}$ $(\tilde{F}_k(\lambda) = \varphi(\lambda)F(2^{-2k}\lambda))$
~~$\sup_y \int dx\, |(\Psi_k F_k)(A)(x,y)|$~~ $F_k(A) = \delta_{2^{-k}}\tilde{F}_k(A_k)\delta_{2^k}$ $\big)$

~~From this $L^1\to L^1$ bound and $\int |F_k(A)(x,y)|(1+|x-y|\cdot 2^{-k})^a dx \lesssim 1$ $(a<\varepsilon)$, we have~~
~~$\forall k,x,y$ $|\Psi_k(A)(x,y)| \lesssim 2^{-kd}(1+2^{-k}|x-y|)^{-d-1}$~~

By dilation and Lemma 3, we obtain $|\Psi_k(A)(x,y)| \leq 2^{-kd}(1+2^{-k}|x-y|)^{-d-1}$
and by combining this with $\int |F_k(A)(x,y)|(1+2^{-k}|x-y|)^a dx \leq C$, we obtain $\|(\Psi_k F_k)(A)\|_{L^1\to L^1} \leq C$

Thus, $\|\sum_i F_{k_i}(A)\Psi_{k_i}(A) b_i\|_{L^1} \lesssim \sum_i \|b_i\|_{L^1} \lesssim \|f\|_{L^1} \Rightarrow$ (C)

$\Uparrow$ (B)

Lemma 4 $\quad \|\sum \psi_{h_i}(A) b_i\|_2^2 \lesssim \lambda \|f\|_{L^1}$

Proof Let $Q := \{x : \max |x_i| \le 1\} \Rightarrow \exists\, c_0$ s.t. $\forall x \in \mathbb{R}^d$, $\sup_{y\in Q} (1+|x-y|)^{-d-1} \le c_0 \inf_{y\in Q} (1+|x-y|)^{-d-1}$

and hence by dilations $\sup_{y\in Q_i} (1+2^{-h_i}|x-y|)^{-d-1} \le c_0 \inf_{y \in Q_i} (1+2^{-h_i}|x-y|)^{-d-1}$

Now fix i and let y_0 be the center of Q_i. By $|\psi_h(A)(x,y)| \le 2^{-hd}(1+|x-y|\cdot 2^{-h})^{-d-1}$ and the preliminary estimate, we have $\qquad \int |b_i(y)| \lesssim \lambda |Q_i|$

$$|\psi_{h_i}(A) b_i|(x) \le \int 2^{-h_i d}(1+2^{-h_i}|x-y|)^{-d-1}|b_i(y)|\,dy$$
$$\lesssim \lambda |Q_i| \cdot 2^{-h_i d}(1+2^{-h_i}|x-y_0|)^{-d-1}$$
$$\lesssim \lambda \int_{Q_i} 2^{-h_i d}(1+2^{-h_i}|x-y|)^{-d-1}\,dy$$
$$= \lambda\,(2^{-h_i d}(1+2^{-h_i}|\cdot|)^{-d-1} * \chi_{Q_i})(x)$$

For some $h \in L^2$, we have $|(h, 2^{-h_i d}(1+2^{-h_i}|\cdot|)^{-d-1} * \chi_{Q_i})|$

$$= |(2^{-h_i d}(1+2^{-h_i}|\cdot|)^{-d-1} * h, \chi_{Q_i})| \lesssim (Mh, \chi_{Q_i})$$

Φ radial, decr., $\int \Phi_t = \int \Phi$

p. 63, Stein 1970 eq ($\sup_{t>0} |(f * \Phi_t)(x)| \le (Mf)(x) \int \Phi\, dy$

III. §2, Thm 2(a)

where M is the standard Hardy-Littlewood maximal operator, i.e., $(Mf)(x) := \sup_{r>0} \frac{c_n}{r^n} \int_{|y|<r} f(x-y)\,dy$

Since M is L^p-bdd (i.p. $p=2$), we have

$$|(h, \sum_i \psi_{h_i}(A) b_i)| \lesssim (Mh, \sum \lambda \chi_{Q_i}) \lesssim \|h\|_2 \underbrace{\|\lambda \sum \chi_{Q_i}\|_2}_{= \lambda\sqrt{\sum |Q_i|} \lesssim \sqrt{\lambda \|f\|_1}} \qquad \blacksquare$$

Thus, (A), (B), and (C) are proven and the weak type (1,1) ineq. follows.