L. Grafakos, Classical Fourier Analysis Chapter 2
E. M. Stein, Sing. integrals Chapter I

2 Maximal functions and covering lemmas

It's well known that the convolution of a ~~fixed~~ function with a fixed density is a smoothing operation that produces a certain average of a function. Averaging is an important operation in analysis and naturally arises in many situations.

The study of averages ~~is that under~~ can be quantified nicely using "the" so-called maximal function (there are a lots of possibilities to define a maximal fct as we shall see.)

Maximal fcts are used to obtain almost everywhere convergence results ~~for~~ (e.g., Bochner-Riesz or Schrödinger evolution, Kakeya ...) (FAP II in Krantz) and more generally differentiation theory of integrals

Although maximal functions do not preserve qualitative information about the given fcts, they do preserve quantitative information.

Recall the following fundamental theorem of Lebesgue which says

$$\lim_{r\to 0} \frac{1}{|B_x(r)|}\int_{B_x(r)} f(y)\,dy = f(x)$$

where $B_x(r)=\{y\in\mathbb{R}^d : |x-y|<r\}$ and $|A|$ denotes the Lebesgue measure of some measurable subset $A\subseteq\mathbb{R}^d$, and $f\in L^1_{loc}(\mathbb{R}^d)$.

In order to study this limit, we consider its quantitative analogue, where "$\lim_{r\to 0}$" is replaced by "$\sup_{r>0}$". Since the properties of this function are expressed in terms of relative size and do not involve any cancellations, we replace f by $|f|$ and define the <u>Hardy-Littlewood maximal function</u>

$$(Mf)(x) := \sup_{r>0}\frac{1}{|B_x(r)|}\int_{B_x(r)} |f(y)|\,dy \quad \left(= \sup_{r>0}\frac{1}{|B_0(r)|}\, f * \mathbb{1}_{B_0(r)}(x)\right)$$

Before we study its properties in detail, let's try to understand its relation with Lebesgue's theorem on a more abstract level.

Afterwards, we'll study its L^p-bddness and generalizations to other geometries where balls are replaced by other objects (such as cubes, rectangles, ...)

Consider again two measure spaces (X,μ), (Y,ν), $0<p\le\infty$, $0<q<\infty$.
Suppose D is a dense ~~set of~~ subspace of $L^p(X,\mu)$ (e.g. C_c^∞, $\mathcal{S}$...) and

T_ϵ is a linear operator defined on $L^p(X,\mu)$ with values in the measurable functions in Y.

Now define the sublinear operator $(T_* f)(x) = \sup_{\epsilon>0} |(T_\epsilon f)(x)|$.

<u>Theorem 2.1</u> Let $0<p<\infty$, $0<q<\infty$, T_ϵ, T_* as above. Suppose that for some $B>0$ and all $f\in L^p$, we have $\|T_* f\|_{L^{q,\infty}} \le B\|f\|_{L^p}$

and for all $f\in D$, $\lim_{\epsilon\to 0} T_\epsilon f = Tf$ exists and is finite ν-a.e. (pointwise)

~~Th~~ Then for all functions $f\in L^p(X,\mu)$, the limit $\lim_{\epsilon\to0} T_\epsilon f \overset{Tf}{=}$ exists and is finite ν-a.e. and defines a linear operator on $L^p(X)$ (uniquely extending T defined on D) that satisfies $\|Tf\|_{L^{q,\infty}} \le B\|f\|_{L^p}$

<u>Proof</u> <u>Exercise</u> (Recall the uniform boundedness principle)

<u>Remark 2.2</u> The uniform bddness principle asserts the following.
Let X be a Banach space and D be a dense subset. Let $T_\epsilon : X\to X$ be a sequence of linear operators (bdd on X) s.t. $T_\epsilon f \xrightarrow[\epsilon\to0]{\|\cdot\|_X} Tf$ in X for all $f\in D$ and some linear operator T that is also bdd on X.
Then, in order to have $T_\epsilon f \xrightarrow[\epsilon\to0]{\|\cdot\|_X} Tf$ for all $f\in X$ it is a necessary and sufficient condition to have the estimate $\|T_\epsilon f\| \lesssim \|f\|$ for all sufficiently small ϵ and all $f\in X$.

E. Stein observed that in Thm 2.1 one does in general <u>not</u> have the converse assertion!

Consider, e.g., $(T_n f)(x) = \int_{\mathbb{R}} 1\!\!1_{[n,n+1)}(x-y) f(y)\,dy$ on $L^1(\mathbb{R})$.

Clearly $(T_n f)(x) \xrightarrow{n\to\infty} 0$ pointwise a.e. for $f\in L^1$ and $\|T_n\|_{L^1\to L^1} \lesssim 1$, uniformly in n. But $T^* f \notin L^{1,\infty}$. However, one does sometimes have sharpness in Thm 2.1. This is known as

e.g. $(\mathbb{R}^d/\mathbb{Z})^d$ or $SO(d)$

<u>Thm 2.2</u> (Stein's maximal principle)
Let G be a compact group, X be a homogeneous space of G with finite Haar measure μ, $1\le p\le 2$, $T_n : L^p(X)\to L^p(X)$ a sequence of bdd linear operators commuting with translations s.t. $T_n f$ converges pt-wise a.e. for each $f\in L^p$.

Then $T^* f \in L^{p,\infty}$.

Examples for T include
- M (maximal operator), $\rightsquigarrow$ Lebesgue differentiation
- $e^{-t\sqrt{-\Delta}}$ (Poisson kernel with $e^{-t\sqrt{-\Delta}} = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \frac{1}{(1+|x|^2)^{(d+1)/2}}$
- $e^{it\Delta}$ (Schrödinger evolution)
- $e^{it(-\Delta+V)}$
- $\left(1-\frac{D^2}{R^2}\right)_+^{\delta}$ (Bochner-Riesz)
- $|T_\omega^\delta(a)|^{-1} \mathbb{1}_{T_\omega^\delta(a)}$, $\omega \in S^{d-1}$, $0<\delta<1$, $a \in \mathbb{R}^d$ (Kakeya)

<u>Theorem 2.3</u> Let f be a given fct on $\mathbb{R}^d$

a) If $f \in L^p$, $1 \le p \le \infty$, then Mf is finite a.e.

b) If $f \in L^1$, then $\|Mf\|_{L^{1,\infty}} \lesssim_n A\|f\|$, but M is not L^1-bdd $\to$ <u>exercise</u> (*)

c) If $f \in L^p$ with $1<p<\infty$, then $\|Mf\|_p \lesssim_n A_p \|f\|_p$

(*) Since $f \not\equiv 0$, there is an $R>0$ s.t. $\cancel{\int_{B_R(0)}}$ $\int_{B_0(R)} |f|\, d\mu \ge \delta > 0$ for some δ.

Now, if $|x|>R$, $\cancel{B_R}$ $B_0(R) \subseteq B_x(2|x|)$ and thus $\cancel{Se}$ $Mf(x) \ge \frac{1}{|B_x(2|x|)|} \int_{B_0(R)} |f| \ge c_d \underbrace{\frac{\delta}{|x|^d}}_{\notin L^1}$

The proof relies on the following covering lemma.

<u>Theorem 2.4</u> (Vitali) (Baby-version)

Let $E \subseteq \mathbb{R}^d$ be a measurable subset which is covered by the union of a family of balls $\{B_j\}$ of bounded diameter. Then, from this family we can select a disjoint subsequence $B_{j_1}, B_{j_2}, \dots, B_{j_k}, \dots$ (finite or infinite) such that

$$\sum_k |B_k| \gtrsim_n |E| \quad \text{(the constant } 5^{-n} \text{ will do, for example.)}$$

<u>Proof</u> In abuse of notation, we will call the balls of the subsequence that we will chose now also B_k. The construction/choice of the B_{j_k} here will be non-unique but this is of no consequence to us.

~~Let B.~~ We chose B_1 so that $\cancel{B_1}$ $\operatorname{diam} B_{j_1} \ge \frac{1}{2} \sup_k \operatorname{diam} B_k$, i.e., essentially as large as possible.

Next, suppose the other $k-1$ balls of the (disjoint) subsequence have been chosen as well, i.e., we are given balls $B_1, \dots B_k$ and next need to select B_{k+1} which belongs to the original family ~~that is~~ of balls ~~of balls~~ which are in addition disjoint with $B_1, \dots B_k$. Again, we choose it as large as possible, i.e., $\cancel{B_{k+1}}$ B_{k+1} is disjoint from $B_1 \dots B_k$ and $\operatorname{diam} B_{k+1} \ge \frac{1}{2} \sup_j \{\operatorname{diam} B_j : B_j \text{ disjoint from } B_1 \dots B_k\}$

In principle, this sequence could terminate at some point which would be finite and be the case if there were no more balls left in the original sequence that are disjoint with the selected $B_1, \ldots B_k$

There are two cases now

(i) $\sum |B_k| = \infty \Rightarrow$ we're done (even if $|E| = \infty$ as well)

(ii) $\sum |B_k| < \infty$

Let B_k^* be the ~~ball with same cen~~ 5-dilate of B_k. We claim

$$E \subset \bigcup_k B_k^* \qquad (*) \quad \left(\text{note that } |\cup B_k^*| \leq 5^d |\cup B_k| = 5^d \textstyle\sum_k |B_k|\right)$$

To prove $(*)$ it suffices to show that every B_j from the ~~seq~~ original sequence of balls is contained in $\bigcup_k B_k^*$. Clearly, we may suppose B_j does not belong to our chosen sequence.

Since $\sum_k |B_k| < \infty$, we ~~then~~ necessarily have $|B_k|$, diam $B_k \xrightarrow{k\to\infty} 0$ and so we take the first k B_k balls with the property that diam $B_{k+1} < \frac{1}{2}$ diam B_j.

Thus, B_j must intersect one of the first B_k balls, say it intersects with B_{j_0} for some $j_0 \in \{1, \ldots, k\}$ which satisfies $\frac{1}{2}$ diam $B_j \leq$ diam B_{j_0}.

But then by an obvious geometric consideration, we necessarily have $B_j \subseteq B_{j_0}^*$ which proves the assertion. $\square$

Alternative version of Vitali (more baby..) Let $\{B_1 .. B_k\}$ be a ~~sequence~~ finite collection of balls in $\mathbb{R}^d$. Then $\exists$ finite subcollection $\{B_{j_1} .. B_{j_\ell}\}$ of pairwise disjoint balls s.t. $\sum_{r=1}^{\ell} |B_{j_r}| \geq 3^{-d} |\bigcup_{i=1}^{k} B_i|$.

We are now in position to give the

Proof of Theorem 2.3

$Mf(x) = \sup_{r>0} \frac{1}{|B_x(r)|} \int_{B_x(r)} |f|$

Let $E_\alpha = \{x : (Mf)(x) > \alpha\}$, i.e., $d_{Mf}(\alpha) = |E_\alpha|$ in the previous notation.

Proof of b) For each $x \in E_\alpha$ there is a ball B_x such that

$$\int_{B_x} |f(y)|\, dy \geq \alpha \cdot |B_x|. \qquad (*)$$

On the one hand, $(*)$ shows $|B_x| \leq \frac{\|f\|_1}{\alpha}$ for all $x \in E_\alpha$.
On the other hand, when x runs through E_α, then the union of the corresponding balls covers E_α, i.e., we can apply Vitali's lemma to extract a subsequence of balls, say $\{B_k\}$, which are pairwise disjoint and satisfy $\sum_{k=0}^{\infty} |B_k| \geq 5^{-d} |E_\alpha| = 5^{-d} d_{Mf}(\alpha)$.

$\Rightarrow$ Combining this with $(*)$ for every B_k of the disjoint collection of balls yields $\|f\|_1 \geq \int_{\bigcup_k B_k} |f| \geq \alpha \sum |B_k| \geq 5^{-d} \cdot \alpha\, d_{Mf}(\alpha)$, $\alpha > 0$

i.e., we've shown $\|Mf\|_{L^{1,\infty}} \leq 5^d \|f\|_{L^1}$. In particular, we obtained part a) of Thm 2.3 when $p = 1$ (i.e., $(Mf)(x) < \infty$ a.e.)

(Simultaneous) proof of a) and c)

$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{\infty} = 1-\theta$

by Marcinkiewicz, $\|Mf\|_p \leq \frac{p \cdot 3^{d/p}}{p-1} \|f\|_p$

$p = \infty$ is trivial with $A_\infty = 1$. (So assume $1 < p < \infty$ from now on.)

We will now see/use a simple technique to split f into its large and small parts (later, we'll do better...)

Let us define $f_1(x)$ by $f_1(x) := \begin{cases} f(x) & \text{if } |f(x)| > \frac{\alpha}{2} \\ 0 & \text{else} \end{cases}$ ~~$\Rightarrow |f(x)| \leq |f_1(x)| + \frac{\alpha}{2}$~~

$\Rightarrow |f(x)| \leq |f_1(x)| + \frac{\alpha}{2}$ and therefore $(Mf)(x) \leq (Mf_1)(x) + \frac{\alpha}{2}$.

~~In particular~~ $\Rightarrow \{x : (Mf)(x) > \alpha\} \subseteq \{x : (Mf_1)(x) > \frac{\alpha}{2}\}$ and finally

$$m(E_\alpha) = d_{Mf}(\alpha) \overset{b)}{\leq} \frac{A}{\alpha/2} \|f_1\|_1 = \frac{2A}{\alpha} \underbrace{\int_{|f| > \alpha/2} |f|\, dx}_{\leq \|f\|_p |\{x : |f(x)| > \frac{\alpha}{2}\}|^{1/p'}}. \qquad (**)$$

$f_1 \in L^1 \Uparrow f \in L^p$

Some remarks are in order.

Remarks 2.5 1) Vitali's lemma can be generalized to other geometric objects such as cubes. The main property that one needs is that if two objects overlap, then the smaller one is contained in a dilate of the other one. This is a fairly generic property, and for instance holds for metric balls on a measure space satisfying the doubling property $\mu(B_x(2r)) = O(\mu(B_x(r)))$ <u>but</u> it fails for very thin or eccentric sets such as long tubes, rectangles, annuli etc! In fact, understanding maximal operators of these types is still a major open problem (see also Kakeya).

2) One can generalize the above $(p=1)$ Hardy-Littlewood maximal function and introduce the p-maximal fct.

$$(M_p f)(x) = \sup_{r>0} (N_{p,r} f)(x), \qquad (N_{p,r} f)(x) = r^{-d/p} \|f\|_{L^p(B_x(r))}$$

This object comes in handy when considering singular integrals which are of weak-type (p,p), instead of $(1,1)$. This object has various nice properties, such as (for $1 \le p \le q \le \infty$)

$$(N_{p,r} f)(x) \le (N_{q,r} f)(x), \qquad \|f\|_p \sim \|N_{p,r}\|_p, \quad \cancel{(N_{q,s} f)(y) \lesssim_{d,q} \left(\tfrac{r}{s}\right)^{d}}$$

$$(N_{q,s} f)(y) \lesssim_{d,q} \left(\frac{r}{s}\right)^{d/q} (N_{q,r} f)(x), \quad x \in \mathbb{R}^d, \ y \in B_x(s), \ r \geq 2s$$

$$\|N_{p,q,r} f\|_q \lesssim_p \|f\|_p \quad \text{where} \quad (N_{pqr} f)(x) = r^{-d/q} \|f\|_{L^p(B_x(r))}$$

See, e.g. Blunck-Kunstmann (2003, 2005)

Rev. Mat. Iberoamericana   J. Oper. Theory

Another useful application of the maximal function ~~is~~ lies in approximation theory and in verifying L^p-bddness of certain operators. Let us first give the relevant result and then present explicit examples (heat-semigroup)

Theorem 2.6 Let $\varphi \in L^1(\mathbb{R}^d)$ and set $\varphi_\epsilon(x) = \epsilon^{-d}\varphi(x/\epsilon)$, $\epsilon > 0$.

Suppose that $\psi(x) := \sup_{|y| \geq |x|} |\varphi(y)|$ is integrable, i.e. $\int_{\mathbb{R}^d} \psi(x) = A < \infty$.

least decreasing radial majorant of φ

Then, with the same A, we have

a) $\sup_{\epsilon > 0} |(f * \varphi_\epsilon)(x)| \leq A\,(Mf)(x)$, $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$

b) If, additionally $\int_{\mathbb{R}^d} \varphi(x)dx = 1$, then $\lim_{\epsilon \to 0} (f * \varphi_\epsilon)(x) = f(x)$ a.e.

c) If $p < \infty$, then $\|f * \varphi_\epsilon - f\|_p \xrightarrow{\epsilon \to 0} 0$

d) If $f \in C_b^0(\mathbb{R}^d)$, then $(f * \varphi_\epsilon)(x) \xrightarrow{\epsilon \to 0} f(x)$ uniformly on compact subsets of $\mathbb{R}^d$.

Examples for φ (Exercise) $\quad \varphi_\epsilon^{(\alpha)}(x) = e^{-\epsilon |p|^\alpha}$ where $|p|^\alpha = (-\Delta)^{\alpha/2}$ in $L^2(\mathbb{R}^d)$ (defined via Plancherel on $H^\alpha(\mathbb{R}^d)$)

$$\varphi_\epsilon^{(2)}(x) = \int_{\mathbb{R}^d} e^{-\epsilon|\xi|^2} e^{2\pi i x\cdot\xi}\, d\xi = \frac{e^{-\pi^2|x|^2/\epsilon}}{(\epsilon/\pi)^{d/2}} \quad \text{(Exercise)}$$

"subordination"

$$\varphi_\epsilon^{(1)}(x) = \int_{\mathbb{R}^d} e^{-2\pi\epsilon|\xi|} e^{2\pi i x\cdot\xi}\, d\xi = C_d \frac{\epsilon}{(\epsilon^2+|x|^2)^{(d+1)/2}}, \quad \cancel{C_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}} \quad C_d = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}$$

(Use and prove: $\quad e^{-\gamma} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\gamma^2/(4u)}\, du$, $\gamma > 0$)

~~and write $e^{-\epsilon|p|}(x)$~~ $=$

$$\varphi_\epsilon^{(\alpha)}(x) \sim \frac{\epsilon}{(|x|^2 + \epsilon^{2/\alpha})^{\frac{d+\alpha}{2}}}$$

(Blumenthal–Getoor (1960) Trans. Amer. Math. Soc.)

(Remark Subordination principle immediately transfers certain properties of one semi-group to the other, for instance monotonicity a.k.a. the maximum principle $\quad$ i.e. $e^{-|p|^2}(x) \leq e^{-|p|^2}(y)$ for $|x| > |y|$

$\Rightarrow e^{-|p|^\alpha}(x) \leq e^{-|p|^\alpha}(y)$ for $|x| > |y|$)

Some remarks on the Poisson integral

$e^{-t|p|^\alpha}$ solves the heat equation $(\partial_t + |p|^\alpha)u = 0$

especially for $\alpha = 1$ the Dirichlet problem on $\mathbb{R}^{d+1}_+ = \{(x,t): x\in\mathbb{R}^d, t>0\}$, i.e., find a harmonic function $u(x,t)$ on $\mathbb{R}^{d+1}_+$ whose boundary values on $\mathbb{R}^d$ are f

i.e. $\begin{cases} \Delta_{x,t} u = \partial_t^2 + \sum_{k=1}^d \partial_{x_k}^2 u = 0 \\ u(x,0) = f \end{cases} \Rightarrow u(x,t) = e^{-t|p|} f = \int e^{-t|\xi|}\hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$

$e^{-t|p|^\alpha}$ (like $e^{-t|p|^2}$ via subordination) has nice properties

(i) $e^{-t|p|^\alpha}(x) > 0$

(ii) $\int e^{-t|p|^\alpha}(x)\,dx = 1$, $t>0$

(iii) $e^{-t|p|^\alpha}$ is homogeneous of degree $-\frac{d}{\alpha}$, i.e., $e^{-t|p|^\alpha}(x) = t^{-d/\alpha} e^{-|p|^\alpha}\left(\frac{x}{t^{1/\alpha}}\right)$

(iv) $e^{-t|p|^\alpha}(x)$ is monotonously decreasing in $|x|$ and $e^{-t|p|^\alpha} \in L^p$, $1\le p\le\infty$

(v) $e^{-t|p|}f$ is harmonic for any $f\in L^p$ $(1\le p\le\infty)$ (since $\int e^{-t|\xi|}e^{2\pi i x\cdot\xi}d\xi$ is harmonic)

(vi) semi-group property $e^{-t|p|^\alpha} * e^{-s|p|^\alpha} = e^{-(t+s)|p|^\alpha}$.

The Poisson integral also appears frequently in spectral theory. Consider a self-adjoint, ~~posi~~ densely defined, linear ~~$\mathbb{R}$~~ operator A in some Hilbert space, say $L^2(\mathbb{R}^d)$ for concreteness. We wish to characterize its spectrum $\sigma(A) = \mathbb{C}\setminus\rho(A)$.

The Borel-Stieltjes transform is predestined for the analysis.

Say $A = \int_{\sigma(A)} \lambda\, dE_A(\lambda)$ is the spectral decomposition and $d\mu_{\psi\psi}(\lambda) \equiv d\mu_\psi(\lambda) = \langle\psi, dE_A(\lambda)\psi\rangle$

where $E_A(\lambda) = E_A((-\infty,\lambda])$. Then we define $F_\psi(z) = \int \frac{d\mu_\psi(\lambda)}{\lambda - z} = \langle\psi, R(z)\psi\rangle$, $R(z) = (A-z)^{-1}$

for $z\in\rho(A)$. Then $\operatorname{Im} F_\psi(z) = \operatorname{Im}(z)\int \frac{d\mu_\psi(\lambda)}{|\lambda - z|^2} = \frac{1}{2i}\left[\langle\psi, R(z)\psi\rangle - \langle\psi, R(\bar z)\psi\rangle\right]$

and in particular $\operatorname{Im} F_\psi(E+i\epsilon) = \int \frac{\epsilon}{(\lambda-E)^2+\epsilon^2}\, d\mu_\psi(\lambda)$.

$\Rightarrow \int \phi(E)\operatorname{Im} F_\psi(E+i\epsilon))\,dE \xrightarrow{\epsilon\to 0} \int \phi(\lambda)\,d\mu_\psi(\lambda)$, $\phi\in C_c(\mathbb{R})$ and one has

Stone's formula $\frac{1}{2}\left(\langle\psi, E(\Lambda)\psi\rangle + \langle\psi, E(\bar\Lambda)\psi\rangle\right) = \ldots = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_\Lambda d\lambda \langle\psi, (R(\lambda+i\epsilon) - R(\lambda - i\epsilon))\psi\rangle$

for all intervals $\Lambda\subseteq\mathbb{R}$. (More interest in that??)

Proof of Thm 2.6

We have already encountered a special case of this situation where $\varphi(x) = \frac{1}{|B_x(r)|} \mathbb{1}_{B_x(r)}$.
The main idea is to reduce matters to this fundamental special case.

$\|f - f*\varphi_\epsilon\|_p \to 0$ $\quad f(x) - f_\epsilon(x) = \int dy\, \varphi_\epsilon(y)\,(f(x) - f(x-y))$

Proof of c) (In fact, the proof holds under the weaker assumption that φ is merely integrable, but still requires the normalization condition $\int \varphi\,dx = 1$)

First, if $f \in L^p(\mathbb{R}^d)$, $p<\infty$, and $\|f(x-y) - f(x)\|_{L^p_x(\mathbb{R}^d)} \equiv \Delta(y)$,
integration wrt x

then $\Delta(y) \to 0$ as $y \to 0$, i.e., the map $\begin{matrix}\mathbb{R}^d \to L^p(\mathbb{R}^d)\\ y \mapsto f(x-y)\end{matrix}$ is continuous. (dom. conv.)

If $f_1 \in C_c^0(\mathbb{R}^d)$, then the assertion immediately follows from the uniform convergence $f_1(x-y) \xrightarrow{y\to 0} f_1(x)$. (remember $f - f*\varphi_\epsilon = \int dy\, \varphi_\epsilon(y)\,(f(x) - f(x-y))$)
For general f write $f = f_1 + f_2$ with f_1 as before and $\|f_2\|_p \le \delta$ for a given fixed δ. (This is possible since such f_1 are dense in L^p if $p<\infty$)
$\Rightarrow \Delta(y) \le \Delta_1(y) + \Delta_2(y)$ with $\Delta_1(y) \xrightarrow{y\to 0} 0$ and $\Delta_2(y) \le 2\delta$ (triangle ineq.)
$\Rightarrow \Delta(y) \xrightarrow{y\to 0} 0$ for general $f \in L^p$, $p<\infty$.
Now for $f*\varphi_\epsilon - f = \int dy\, \varphi_\epsilon(y)\,(f(x-y) - f(y))$ we have

$$\|f*\varphi_\epsilon - f\|_p \le \int \Delta(y)\,|\varphi_\epsilon(y)|\,dy = \int \Delta(\epsilon y)\,|\varphi(y)|\,dy \to 0 \text{ by dom. conv.}$$

$|(f*\varphi_\epsilon)(x)| \le A\,(Mf)(x)$, $\epsilon>0$

Proof of a) Let us write $\psi(r) \equiv \psi(x)$ for $r = |x|$ (ψ is radial)
Observe that $\int_{\frac{r}{2}\le |x|\le r} \psi(x) \ge \psi(r)\cdot c\cdot r^d$. Thus, since $\psi \in L^1$ and ψ decreases,
we have $r^d \cdot \psi(r) \to 0$ as $r \to 0$ or $r \to \infty$.
(dom conv.)
To prove a), it suffices to show $|(f*\psi_\epsilon)(x)| \le A\,(Mf)(x)$ with $f \ge 0$, $1 \le p \le \infty$, $A = \int_{\mathbb{R}^d} \psi$
Since the assertion is translational invariant (wrt f)
and dilation invariant (wrt ψ) it suffices to prove $(f*\psi)(0) \le A\,(Mf)(0)$
and we may assume $(Mf)(0) < \infty$.
Let $\lambda(r) = \int_{S^{d-1}} f(r\omega)\,d\omega$ and $\Lambda(r) = \int_{|x|<r} f(x)\,dx$, i.e., $\Lambda(r) = \int_0^r \lambda(t)\cdot t^{d-1}\,dt$
(spherical average) (ball average)

We have

$$(f * \Psi)(0) = \int f(x)\Psi(x)dx = \int_0^\infty dr\ r^{d-1} \lambda(r) \Psi(r)$$

$$= \lim_{\substack{\epsilon\to 0\\ N\to\infty}} \int_\epsilon^N dr\ \underbrace{r^{d-1}\lambda(r)}_{\Lambda'(r)}\Psi(r)$$

$\Lambda(r) = \int_0^r dt\ t^{d-1}\lambda(t)$

$\Lambda'(r) = r^{d-1}\lambda(r) - 0$

$$\overset{I.B.P.}{=} - \lim_{\substack{\epsilon\to 0\\ N\to\infty}} \int_\epsilon^N \Lambda(r)\, d\Psi(r)$$

using that the boundary term $\Lambda(N)\Psi(N) - \Lambda(\epsilon)\Psi(\epsilon) \xrightarrow[\epsilon\to 0]{N\to\infty} 0$ bc of $r^d\Psi(r) \to 0$ and $\Lambda(r) \le |S|^{d-1} r^d (Mf)(0)$

$\le \Lambda(N)\Psi(N) + \Lambda(\epsilon)\Psi(\epsilon)$

$\le (Mf)(0)\underbrace{[N^d\Psi(N) + \epsilon^N\Psi(\epsilon)]}_{\to 0.}$

$$\text{Therefore } (f*\Psi)(0) = \int_0^\infty \underbrace{\Lambda(r)}_{\le |S|^{d-1} r^d} d(-\Psi(r)) \le |S|^{d-1} (Mf)(0) \underbrace{\int_0^\infty r^d\, d(-\Psi(r))}_{= A/|S^{d-1}|}$$

$$= A\,(Mf)(0)$$

~~Proof of c) [illegible] Exercise ($p<\infty$: similar as in pf of Thm 2.3~~

~~split $f = f_1 + f_2$~~ $\quad$ C_c^0 $\quad$ ~~small~~ L^p-norm

~~+ use FAP~~

~~$p=\infty$~~

Proof of c) $p<\infty$: follows from FAP (Thm 2.1) and that the a.e.-convergence clearly holds for C_c^0 - fcts which are dense in L^p, $p<\infty$

$p=\infty$: Exercise (see, e.g. Stein - Singular Integrals p. 64)

Proof of d) Let $V \subset\subset \mathbb{R}^d$ and chose $W \subseteq \mathbb{R}^d$ s.t. $V \subset\subset W \subset\subset \mathbb{R}^d$. Then, $f|_W$ is uniformly continuous (since $f \in C_b^0$) $\Rightarrow$ the assertion follows from b) since the assertion there holds uniformly for $x \in V$. $\square$

We close this chapter with another covering lemma, which, however, does not involve measure theory but deals with the geometric structure of general closed sets F in $\mathbb{R}^d$: can the complement F^c be realized as a disjoint union of certain cubes in a "canonical" way? For $d=1$, the answer is of course yes since every open set is in a unique way the union of disjoint open intervals. For $d \geq 2$ the situation is not anymore so simple since arbitrary open sets can be realized in infinitely many ways by disjoint unions of cubes (by cubes, we mean closed cubes and by disjoint we mean that their interiors are disjoint)

The following lemma by Whitney is, while not canonical, useful in many applications (e.g. bilinear restriction theory..)

Definition 2.7 A dyadic cube in $\mathbb{R}^d$ of generation n is a set of the form

$$Q = Q_{n,k} = 2^n\left(k + [0,1)^d\right) = \{2^n(k+x) : x \in [0,1)^d\}, \quad n \in \mathbb{Z}, \ k \in \mathbb{Z}^d$$

simpler, as there is only one relevant dimension

The crucial property of dyadic cubes k (or dyadic annuli, ...) is their nesting property: If two dyadic cubes overlap, then one must, in fact, contain the other.

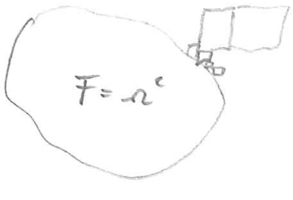
Proposition 2.8 (Dyadic Whitney decomposition I)
Let $\Omega \subsetneq \mathbb{R}^d_+$ be an open set. Then there exists a decomposition $\Omega = \bigcup_{Q \in \mathcal{Q}} Q$ where Q ranges over a family $\mathcal{Q}$ of disjoint dyadic cubes and for each cube $Q \in \mathcal{Q}$, the parent Q' of Q is not contained in Ω.
(the unique dyadic cube of twice the side length containing Q

Proof Define $\mathcal{Q}$ to be the set of all dyadic cubes in Ω which are maximal wrt set inclusion; the claim follows from the nesting property. (The condition $\Omega \subsetneq \mathbb{R}^d_+$ is needed to ensure that every cube is contained in a maximal cube; the open-ness is to get every point of Ω contained in at least one cube) $\square$

$(A,B) \leq (A',B')$
$\Updownarrow$
$A \subseteq A'$ and $B \subseteq B'$

The property that Q' (the parent) is not contained in Ω implies the bounds $0 \le \operatorname{dist}(Q, \mathbb{R}^d_+ \setminus \Omega) \le \operatorname{diam} Q$.

(Suppose Q was farther away $\Rightarrow$ its parent could be still packed into Ω ↯)



In many applications one needs to complement the upper bound with a non-trivial lower bound

<u>Proposition 2.9</u> (Dyadic Whitney decomposition II)
Let $\Omega \subsetneq \mathbb{R}^d$ be open and $K \ge 1$. Then there exists a decomposition ~~s.t.~~ $\Omega = \bigcup_{Q \in \mathcal{Q}} Q$ where Q ranges over a family of disjoint dyadic cubes $\mathcal{Q}$ and for each Q, it holds that $\operatorname{dist}(Q, \mathbb{R}^d \setminus \Omega) \sim K \cdot \operatorname{diam}(Q)$

<u>Proof</u> Let $\mathcal{Q}'$ denote those dyadic cubes in Ω so that ~~$K \operatorname{diam} Q \le$~~

$$K \operatorname{diam} Q \le \operatorname{dist}(Q, \Omega^c) \le 5K \operatorname{diam} Q.$$

Then these cubes cover Ω; indeed, for any given $x \in \Omega$, all one needs to do is to locate a cube Q containing x whose diameter satisfies $\frac{\operatorname{dist}(x, \Omega^c)}{4K} \le \operatorname{diam} Q \le \frac{\operatorname{dist}(x, \Omega^c)}{2K}$

The cubes are not disjoint; however, if one lets $\mathcal{Q} \subseteq \mathcal{Q}'$ to be those cubes in $\mathcal{Q}'$ which are maximal wrt set inclusion, then the claim follows from the nesting property. $\boxtimes$

<u>Remark</u> If K is large, then the cubes in the above decomposition have the property that nearby cubes Q, Q' (in the sense that $\operatorname{dist}(Q, Q') \lesssim \operatorname{diam} Q + \operatorname{diam} Q'$) have comparable diameter (bc of the triangle inequality)

$$|\operatorname{dist}(Q, \Omega^c) - \operatorname{dist}(Q', \Omega^c)| \le \operatorname{dist}(Q, Q') + \operatorname{diam} Q + \operatorname{diam} Q'$$

Ex

Since every cube is contained in a ball of comparable radius (with constants only depending on d), we also have

Proposition 2.10 (Whitney decomposition for balls)

Let $\Omega \subsetneq \mathbb{R}^d$ be open and $K \geq 1$. Then one can cover Ω by balls B such that $\operatorname{dist}(\Omega^c, B) \sim_d K \cdot \operatorname{rad}(B)$ and such that each point in Ω is contained in at most $O_d(1)$ balls.

Theorem 2.11 (Whitney decomposition - nondyadic version)

Let $\emptyset \neq F \subseteq \mathbb{R}^d$ closed. Then its complement Ω is the union of sequence of cubes Q_k whose sides are parallel to the axis, whose interiors are mutually disjoint and whose diameters $\sim$ distance from F, i.e,

(i) $\Omega = F^c = \bigcup_{k=1}^{\infty} Q_k$

(ii) $Q_j^{\circ} \cap Q_k^{\circ} = \emptyset$ if $j \neq k$

interior

(iii) $\exists\, c_1, c_2 > 0$ (e.g. $c_1 = 1$, $c_2 = 4$) such that

$$c_1 \operatorname{diam} Q_k \leq \operatorname{dist}(Q_k, F) \leq c_4 \operatorname{diam} Q_k$$

Proof. See, e.g., Stein - Singular Integrals, Chapter VI
or Grafakos - Classical Fourier Analysis, Appendix J

Decomposition of open sets in $\mathbb{R}^d$ into cubes

~~The decomposition of a given set in a disjoint union of cubes (or balls) is a fundamental tool (Vitali) in the study of "geometric" maximal functions~~

~~→ now we consider the related problem which however does not involve measure theory but deals with the geometric structure of open closed sets F in $\mathbb{R}^d$~~

~~Q: Can the complement F^c~~

Another approach to the theory of maximal functions and singular integrals in the next chapter uses the following decomposition lemma of Calderón and Zygmund.

Theorem 2.12 Let $0 \leq f \in L^1(\mathbb{R}^d)$ and $\alpha > 0$ a height. Then there exists a decomposition of $\mathbb{R}^d$ so that

(i) $\mathbb{R}^d = F \cup \Omega$ where $\Omega \cap F = \emptyset$

(ii) $f(x) \leq \alpha$ on F

(iii) $\Omega = \bigcup_k Q_k$ is the union of cubes whose interior is disjoint and so that for each Q_k we have the estimates on $\mathrm{Avg}_{Q_k} f$

$$\alpha \leq \frac{1}{|Q_k|} \int_{Q_k} f \, dx \leq 2^d \alpha \qquad (*)$$

Corollary 2.13 Suppose f, α, F, Ω, and Q_k have the same meaning as in Thm 2.12. Then, there exist constants $A = A_d$, $B = B_d$ so that (i) and (ii) hold and

will be used in next chapter

a) $|\Omega| \leq \frac{A}{\alpha} \|f\|_{L^1}$

b) $\frac{1}{|Q_k|} \int_{Q_k} f \leq B\alpha$; (Proof: in fact, by $(*)$ we can take $B = 2^d$ and, by $(*)$, we indeed have $|\Omega| = \sum_k |Q_k| \leq \frac{1}{\alpha} \int_\Omega |f| \leq \frac{1}{\alpha} \|f\|_{L^1}$)

In fact, the Q_k are at a distance from F comparable to their diameters (by an application of Whitney, as we'll see in a moment)

Remark Sometimes one writes the CZ-lemma in the form

$$f = g + b\,\cancel{\sum \chi_k}, \quad \cancel{\operatorname{supp} b_k \subset Q_k}$$

$$g = \begin{cases} f & x \in F \\ |Q_k|^{-1} \int_{Q_k} f(x)\,dx = \text{const} & x \in Q_k \end{cases} \Rightarrow |g| \lesssim \alpha \quad \text{and} \quad b_k = f\chi_{Q_k}$$

$$b = \sum b_k$$

$$\int b_k = 0 \text{ for all } k$$

Before we prove Thm 2.12, let's give an alternative, more indirect proof of Corollary 2.13 using the $L^1 \to L^{1,\infty}$-bddness of M + ~~Vitali~~ Whitney. This will further clarify the roles of the sets F and Ω into which $\mathbb{R}^d$ was divided.

Alternative proof of Corollary 2.13

Although we know (from Thm 2.12) that $f \le \alpha$ in F, this is not what determines F.

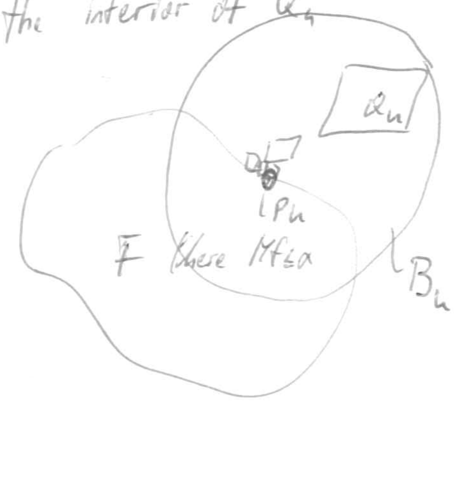
In effect, F is determined by the fact that $Mf(x) < \alpha$ on F.

So we choose $F \equiv \{x : Mf(x) \le \alpha\}$ and $\Omega = E_\alpha = \{x : (Mf)(x) > \alpha\}$

(recall $f(x) = \lim_{R\to 0} \frac{1}{R^d}\int_{B_x(R)} f \le \sup_{R>0} \frac{1}{R^d}\int_{B_x(R)} f = Mf \le \alpha$) $\Rightarrow$ (ii) $\|Mf\|_{1,\infty} \le 5^d\|f\|_1$ says $|\Omega| \le \frac{5^d}{\alpha}\|f\|_{L^1}$ a)

Since F is closed, we can apply Whitney (Thm 2.11) (i) to decompose $\Omega = \bigcup_k Q_k$ where $\operatorname{diam} Q_k \sim \operatorname{dist}(Q_k, F)$. Let Q_k be one of these cubes and $p_k \in F$ a point such that ~~$\operatorname{dist}(F,Q_k) = \operatorname{dist}(p_k,Q_k)$~~ $\operatorname{dist}(F, Q_k) = \operatorname{dist}(p_k, Q_k)$

Now let B_k be ~~a ball~~ the smallest ball whose center is p_k and which contains the interior of Q_k.



Let $\gamma_k = \frac{|B_k|}{|Q_k|} \ge 1$; since $p_k \in F$, we have

$\gamma_k \le$ const, $\operatorname{diam} Q_k \sim \operatorname{dist} Q_k, p_k$

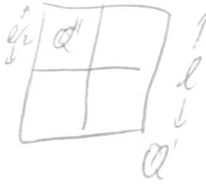
$$\alpha \ge Mf(p_k) \underset{\text{Def of } M}{\ge} \frac{1}{|B_k|}\int_{B_k} f \ge \frac{1}{\gamma_k |Q_k|}\int_{Q_k} f$$

$$\gtrsim \frac{1}{|Q_k|}\int_{Q_k} f,$$ thereby showing b)

Pf of Thm 2.12 ("stopping time argument)

Decompose $\mathbb{R}^d$ into a mesh of ~~cubes~~ equal cubes whose interiors are disjoint and whose common diameter is so large that $\frac{1}{|Q'|}\int_{Q'} f \le \alpha$ for every Q' in our given mesh.

Let Q' be a fixed cube of our given mesh and divide it into 2^d congruent cubes by bisecting each of the sides of Q'.



$|Q'| = 2^d |Q''|$

Let Q'' be one of the "childs" of Q' with half side length.

For this Q'', we have two cases

(i) $\frac{1}{|Q''|}\int_{Q''} f \le \alpha$

(ii) $\frac{1}{|Q''|}\int_{Q''} f > \alpha$ → don't divide Q'' any further and Q'' is selected as one of the cubes Q_k appearing in the statement of the theorem

→ For this Q_k we have the claimed estimate (*) in (ii), because

$$\alpha < \frac{1}{|Q''|}\int_{Q''} f \le \frac{1}{2^{-d}|Q'|}\int_{Q'} f \le 2^d \alpha \quad ✓$$

Remains to treat the case (1) → just proceed in dividing the cubes until (if ever) we are forced into case (2)

→ Denote by $\Omega = \bigcup_k Q_k$ the union of cubes obtained from case (2) where we started the process with all possible cubes Q' of our initial mesh.

We're thus left to prove $f(x) \le \alpha$ whenever $x \in F = \Omega^c$ (a.e.).

But because of Lebesgue differentiation for cubes (in particular dyadic cubes) we have $f(x) = \lim_{Q \to \{x\}} \frac{1}{|Q|}\int_Q f(y)\,dy$ where "$\lim_{Q\to\{x\}}$" is taken over all cubes containing x and the diameter goes to zero. But each of the cubes that enter ~~the~~ in our decomposition that contain $x \in F$, is a cube where case (i) holds, i.e. $\frac{1}{|Q'|}\int_{Q'} |f| < \alpha$, so taking the limit, this shows $f < \alpha$ in F $\boxtimes$

Exercise (Blunck–Kunstmann – CZ theory for non-integral operators and the H^∞ functional calculus 2003)

Thm 3.1

(CZ decomposition for L^p fcts)

~~Let Ω~~

Let $p \in [1,\infty)$. Then there is a constant $A>0$ s.t for all $f \in L^p(\mathbb{R}^d)$, $\alpha>0$, there's a function g and a function sequence $(b_k)_{k\in\mathbb{N}}$ and balls B_k^* (containing the old Q_k) s.t

(i) $f = g + \sum b_j$

(ii) $\|g\|_p \leq C\alpha$

(iii) $\operatorname{supp} b_j \subseteq B_j^*$ and $\#\{k : x \in B_k^*\} \leq A$ for all $x \in \mathbb{R}^d$

(iv) $\|b_k\|_{L^p} \leq C\alpha\, |B_k^*|^{1/p}$

(vi) $\left(\sum_k |B_k^*|\right)^{1/p} \leq C \dfrac{\|f\|_p}{\alpha}$.

$$\left(\sum_k |B_k^*|\right)^{1/p} = \Big\|\sum \mathbb{1}_{B_k^*}\Big\|_p$$