Harmonic Analysis
Homework Sheet 6

Exercise 6.1
Show the following Littlewood principle.

Lemma 0.1. *Let* $1 \leq q, p < \infty$ *and* T *be a non-zero translation invariant operator on* $\mathbb{R}^d$*. Then the estimate* $\|Tf\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$ *is only possible for* $q \geq p$*.*

Exercise 6.2
Let $a \in \mathbb{R} \setminus \{0\}$ and $0 < \varepsilon < \eta < \infty$. Show that

$$|\int_\varepsilon^\eta \frac{\cos(ar) - \cos r}{r}\, dr| \leq 2|\log \frac{1}{|a|}|\,,$$

$$\lim_{\varepsilon \to 0,\, \eta \to \infty} \int_\varepsilon^\eta \frac{\cos(ar) - \cos r}{r}\, dr = \log \frac{1}{|a|}\,.$$

Exercise 6.3
Let $\xi \in \mathbb{R}^d \setminus \{0\}$. Recalling $|\mathbb{S}^{d-1}| = 2\pi^{d/2}\Gamma(d/2)$, show that

$$\int_{\mathbb{S}^{d-1}} \operatorname{sgn}(\xi \cdot \omega)\omega_j \, d\omega = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} \frac{\xi_j}{|\xi|}\,.$$

For $j \in \{1, ..., d\}$ let

$$(R_j f)(x) := (W_j * f)(x) := \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \text{p. v.} \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy\,.$$

be the j-th Riesz transform of $f \in \mathcal{S}(\mathbb{R}^d)$. Using the above identity, show that R_j is given in Fourier space by multiplication with $-i\xi_j|\xi|^{-1}$, i.e., for $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$(R_j f)(x) = (-\frac{i\xi_j}{|\xi|}\hat{f}(\xi))^\vee(x)\,.$$

Show that the Riesz transforms satisfy

$$-1 = \sum_{j=1}^d R_j^2 \quad \text{on } L^2(\mathbb{R}^d)$$

where -1 is understood as the identity operator. For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq j, k \leq d$ show that

$$\partial_{x_j}\partial_{x_k}\varphi(x) = -R_j R_k \Delta\varphi(x)\,, \quad x \in \mathbb{R}^d\,.$$

Exercise 6.4

Show that the Dini-type condition on $\Omega(x)$

$$\int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta < \infty \quad \text{with } \omega(\delta) := \sup_{|x-x'|\le\delta,\, |x|=|x'|=1} |\Omega(x) - \Omega(x')|$$

in Theorem 3.11 implies Hörmander's condition

$$\sup_{y,z\in\mathbb{R}^d} \int_{|x-z|\ge 2|y-z|} |K(x-z) - K(x-y)| \lesssim 1 \quad \text{for } K(x) = \frac{\Omega(x)}{|x|^d}\,.$$

Moreover, establish c) in Theorem 3.12 in the notes.