Harmonic Analysis (Summer term 2020)

December 13, 2019

Lectures: Wed 16.45-18.15 (in F 513) and Thu 8.00-9.30 (in F 315), starting on April 15, 2020

Exercises: Thu 9.45-11.15 (in F 315)

Analysis in general tends to revolve around the study of general classes of *functions* and *operators*. *Real-variable harmonic analysis* focuses in particular on the relation between *qualitative* properties (such as measurability, boundedness, differentiability, analyticity, integrability, decay at infinity, convergence, etc.) and their *quantification* (i.e., what is the smallest upper bound on a function, how often is it differentiable, what is its L^p norm, what is the convergence rate of a sequence, etc.). It is then natural to ask how quantitative properties of such functions change when one applies various (often quite explicit) operators. It turns out that quantitative estimates, such as L^p estimates on such operators, provide an important route to establish qualitative results and in fact there are a number of principles (such as the uniform boundedness principle or Stein's maximal principle [29]) which assert that this is the *only* route, in the sense that a quantitative result must exist in order for the qualitative result to be true.

Many arguments in harmonic analysis will, at some point, involve a combinatorial statement about certain types of geometric objects such as cubes, balls, or tubes. One such useful statement is the *Vitali* covering lemma which asserts that given a collection of balls $B_1, ..., B_k$ in Euclidean space, then there exists a subcollection of balls $B_{i_1}, ..., B_{i_m}$ which are disjoint but contain a significant fraction of the volume covered by the original balls, in the sense that $|\bigcup_{i=1}^k B_k| \leq a_d |\bigcup_{j=1}^m B_{i_j}|$ for some d-dependent constant a_d.

One feature of harmonic analysis methods is that they tend to be *local* rather than *global*. For instance, it is quite common to analyze a function f by applying cutoff functions in either the spatial or the frequency variables to *decompose f into a number of somewhat localized pieces.* One then estimates each of these pieces separately and "glues" the estimates back together at the end. One reason for this "divide and conquer" strategy is that generic functions tend to have infinitely many degrees of freedom (f may for instance be very smooth but slowly decaying at one place whereas at other places f may be highly singular or oscillating very quickly) and it would be quite difficult to treat all of these features at once. A well chosen decomposition can isolate these features from each other, so that each component only has one salient feature that could cause difficulty. In reassembling the estimates from the individual components, one can use rather crude tools such as the triangle inequality, or more refined tools, such as ones relying on *(almost) orthogonality*. The main drawback of decomposition methods is however that one generally does not obtain the optimal constants.

Another basic theme of harmonic analysis is the attempt to quantify the elusive phenomenon of *oscillation*. Intuitively, if an expression oscillates wildly in phase, then its average value should be relatively small in amplitude. This leads to the *principle of stationary phase* and the *Heisenberg uncertainty principle* which relates the decay and smoothness of a function to the smoothness and the decay of its Fourier transform. The development of a robust theory for *oscillatory integrals* is also one main ingredient to understand the interplay between L^p estimates of certain Fourier multipliers (such as the *Bochner–Riesz means*) and geometric properties of certain (smooth) manifolds, such as the Fourier transform of the associated surface measure and L^p estimates for it.

1. Interpolation theorems (Marcinkiewicz (in particular restricted weak-type formulation, see, e.g., Tao [34, Lecture 2]), Riesz–Thorin, Stein [33])

2. Covering lemmas (Vitali, Whitney, Calderón–Zygmund) as well as Calderón–Zygmund decomposition following Stein [30, Chapter 1], see also Grafakos [17, Proposition 2.1.20, Theorem 4.3.1], and Guzmán [11]

3. Maximal functions following Stein [30, 32]

   (a) Hardy–Littlewood maximal function and Lebesgue differentiation theorem following Stein [30, Chapter 1], [32, Chapter I, Section §3]

   (b) Maximal functions and Lebesgue differentiation for more general sets (instead of balls), see Guzmán [11], in particular Cordoba–Fefferman [9]

   (c) Hardy–Littlewood p maximal function (see Blunck–Kunstmann [3, 1])

   (d) Relation to convergence almost everywhere, first glance at Bochner–Riesz summability (FAP1 and FAP2 in [22]. In this regard, see also the "ergodic Hopf–Dunford–Schwartz" theorem [31, p. 48] respectively Dunford–Schwartz [12] (Section XIII.6: Lemma 7 (p. 676), Theorem 8 (p. 678); Section XIII.8: Lemma 6 (p. 690) Theorem 7 (p. 693); Section XIII.9: Exercise 3 (p. 717)))

4. Singular integrals following Stein [30, Chapter II], [32, Chapter I, Section §5]. In particular, Hilbert transform and its application to partial sums operators [30, Chapter IV, Section §4] and second glance at Bochner–Riesz (box multiplier versus disc multiplier, see also Fefferman [15, 16])

5. Riesz transforms and Poisson integrals following Stein [30, Chapter III] and [31, Section §4.4]

6. A primer on the Fourier transform (Wolff [38, Chapters 1-5]) and Mikhlin–Hörmander multiplier theorem for Fourier multipliers [32, Chapter VI, Section §4.4] (see also Sogge [28, Theorem 0.2.6]) and square function estimates / Littlewood–Paley inequalities [23] (rough version with dyadic cubes as in Stein [30, pp. 103-108] or Duoandikoetxea [13, Theorem 8.4] or with dyadic annuli as in Tao [37, Lecture 2, Theorem 1] or arbitrary intervals (when all intervals have the same length, the result is sharp, see Carleson [8], Córdoba [10], and Rubio de Francia [25]) as in Rubio de Francia [26]; smooth version using bump functions or heat kernels as in Killip et al [21, Theorem 4.3] or [20, Theorem 5.3]). Generalization to general self-adjoint operators (such as Schrödinger operators) instead of mere Fourier multipliers: spectral multiplier theorems, Bernstein estimates, and Littlewood–Paley inequalities and their application in nonlinear PDEs

7. Introduction to pseudodifferential operators following Stein [32, Chapter VI] (see also Martinez [24] (Chapter 2, in particular from Section 2.5 on))

   (a) Symbolic calculus, composition [32, Chapter VI, Section §3] and [24, Sections 2.6, 2.7]

   (b) L^2 boundedness, Calderón–Vaillancourt theorem [32, Chapter VI, Section §2] and [24, Section 2.8]

   (c) Singular integral representation, bounds on integral kernels [32, Chapter VI, Section §4.1-4.3]

   (d) L^2 boundedness of translation invariant Calderón–Zygmund operators [32, Chapter VI, Section §4.5]

   (e) Estimates in L^p, Sobolev, and Lipschitz spaces [32, Chapter VI, Section §5]

8. More on (spectral) multiplier theorems. See in particular Hebisch [18], Duong–Ouhabaz–Sikora [14], Blunck–Kunstmann [3], and Blunck [2]

9. Almost orthogonality following Stein [32, Chapter VII]

   (a) Exotic and forbidden symbols, failure of L^2 boundedness for symbols in $S^0_{1,1}$ [32, Chapter VII, Section §1]

   (b) Cotlar–Stein lemma [32, Chapter VII, Section §2.1-2.3] and generalization to Schatten classes, see Carbery [7]

   (c) Consequences of Cotlar–Stein for symbols in $S^0_{\rho,\rho}$ (with $0 \leq \rho < 1$) [32, Chapter VII, Section §2.4-2.5]

   (d) L^2 theory for Calderón–Zygmund operators [32, Chapter VII, Section §3]

   (e) More on the Cauchy integral [32, Chapter VII, Section §4]

10. Uncertainty principle following Wolff [38, Chapter 5]

11. Oscillatory integrals following Stein [32, Chapter XIII, IX]

(a) Oscillatory integrals of the first kind, stationary phase [32, Chapter XIII, Section §1-2]

(b) Fourier transform of surface measures [32, Chapter XIII, Section §3] and application to the lattice counting problem (improvement of Weyl's law for $-\Delta$ on $\mathbb{T}^d$) (Sogge [27, pp. 83-85])

(c) Introduction to Fourier restriction [32, Chapter XIII, Section §4]

(d) Oscillatory integrals of the second kind, Carleson–Sjölin and Hörmander integral operators [32, Chapter IX, Section §1], see also Bourgain [4]

(e) Relation to Fourier restriction and Bochner–Riesz summability [32, Chapter IX, Section §2], Sogge [28, Sections 2.2-2.3], Tao [34, 35, 36]

12. Decoupling inequalities [5, 6, 19]

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