Continuum Calogero-Moser Systems: Recent Results & Open Questions

Enno Lenzmann

Department of Mathematics and Computer Science
University of Basel, Switzerland

Gauss Mini Workshop on Analysis, PDE, and Mathematical Physics
TU Braunschweig, Germany
January 2025

Contents of Talk

This talk deals with **continuum Calogero-Moser (CM) systems**, which is a novel class of **completely integrable PDEs** that typically exhibit **turbulent solutions**.

Agenda:

1. What are Continuum Calogero-Moser Systems?
2. Half-Wave Maps Equation as Pathfinder
3. Recent Results*
4. Outlook & Open Questions

*joint work with **Patrick Gérard** (Paris-Saclay)

Introduction: Genesis of Calogero-Moser Systems

We begin in 1975 with a seminal work by **Jürgen Moser** in (Adv. Math. 1975)

ADVANCES IN MATHEMATICS **16**, 197–220 (1975)

Three Integrable Hamiltonian Systems
Connected with Isospectral Deformations*

J. MOSER

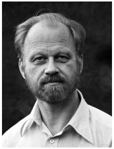
Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

DEDICATED TO STAN ULAM

1. INTRODUCTION

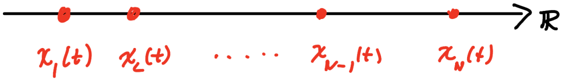
(a) *Background.* In the early stages of classical mechanics it was the ultimate goal to integrate the differential equations of motions explicitly or by quadrature. This led to the discovery of various "integrable" systems, such as Euler's two fixed center problems, Jacobi's integration of the geodesics on a three-axial ellipsoid, S. Kovalevski's motion of the top under gravity for special ratios of the principal moments of inertia, to name a few nontrivial examples. These efforts and their climax with the work of Jacobi who applied skillfully the method of separation of variables to partial differential equations, the Hamilton–Jacobi equations associated with the mechanical system, to establish their integrable character.

However, this development took a sharp turn when Poincaré showed that most Hamiltonian systems are not integrable and gave arguments



- Recent breakthroughs due to Gardner et al. (1967) and P. Lax (1968) on complete integrabilty of Korteweg-de Vries equation (KdV).
- Moser's idea: Use Lax pairs for Hamiltonian systems with **finitely many degrees of freedom** to solve a conjecture by F. Calogero (1971).

Classical Calogero-Moser Systems



Jürgen Moser (Adv. Math. 1975) proved **complete integrability** of the classical N-body system with Hamiltonian

$$\boxed{H_N = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \frac{1}{2}\sum_{k\neq\ell}^{N} \frac{1}{(x_k - x_\ell)^2}}$$

with positions $x_k \in \mathbb{R}$ and momenta $p_k \in \mathbb{R}$ for $1 \leq k \leq N$.

Classical Calogero-Moser Systems

Jürgen Moser (Adv. Math. 1975) proved **complete integrability** of the classical N-body system with Hamiltonian

$$\boxed{H_N = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \frac{1}{2}\sum_{k\neq\ell}^{N} \frac{1}{(x_k - x_\ell)^2}}$$

with positions $x_k \in \mathbb{R}$ and momenta $p_k \in \mathbb{R}$ for $1 \leq k \leq N$.

- Lax–Moser matrix $\mathbf{L}(t) = \mathbf{L}(\vec{x}(t), \vec{p}(t)) \in \mathbb{C}^{N\times N}$ yields set of conserved quantities
$$I_k = \mathrm{Tr}(\mathbf{L}^k(t)) = \text{const.} \quad \text{for } k = 1, \ldots, N$$
- Involution $\{I_k, I_l\} = 0$ yields complete integrability (in Poincaré sense)

Classical Calogero-Moser Systems

The completely integrable **Calogero-Moser (CM) Hamiltonian**:

$$\boxed{H_N = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \frac{1}{2}\sum_{k\neq\ell}^{N} \frac{1}{(x_k - x_\ell)^2}}$$

with positions $x_k \in \mathbb{R}$ and momenta $p_k \in \mathbb{R}$ for $1 \leq k \leq N$.

- Olshanetsky-Perelomov (Invent. Math. 1976) found an alternative proof using **free matrix flows**, i. e., consider the Hermitian matrix

$$\boxed{\mathbf{X}(t) = \mathbf{X}_0 + t\mathbf{L}_0}$$

where $\mathbf{L}_0 =$ Moser's $\mathbf{L}|_{t=0}$ and $\mathbf{X}_0 = \mathrm{diag}(x_1(0), \ldots, x_N(0))$ (initial positions).

Classical Calogero-Moser Systems

The completely integrable **Calogero-Moser (CM) Hamiltonian**:

$$\boxed{H_N = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \frac{1}{2}\sum_{k\neq\ell}^{N} \frac{1}{(x_k - x_\ell)^2}}$$

with positions $x_k \in \mathbb{R}$ and momenta $p_k \in \mathbb{R}$ for $1 \leq k \leq N$.

- Olshanetsky-Perelomov (Invent. Math. 1976) found an alternative proof using **free matrix flows**, i. e., consider the Hermitian matrix

$$\boxed{\mathbf{X}(t) = \mathbf{X}_0 + t\mathbf{L}_0}$$

  where $\mathbf{L}_0 =$ Moser's $\mathbf{L}|_{t=0}$ and $\mathbf{X}_0 = \mathrm{diag}(x_1(0), \ldots, x_N(0))$ (initial positions).
- **Eigenvalues** $\{x_k(t)\}_{k=1}^N$ of matrix $\mathbf{X}(t)$ **solve CM-system**.

Classical Calogero-Moser Systems

The completely integrable **Calogero-Moser (CM) Hamiltonian**:

$$\boxed{H_N = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \frac{1}{2}\sum_{k\neq\ell}^{N} \frac{1}{(x_k - x_\ell)^2}}$$

with positions $x_k \in \mathbb{R}$ and momenta $p_k \in \mathbb{R}$ for $1 \leq k \leq N$.

- Olshanetsky-Perelomov (Invent. Math. 1976) found an alternative proof using **free matrix flows**, i. e., consider the Hermitian matrix

$$\boxed{\mathbf{X}(t) = \mathbf{X}_0 + t\mathbf{L}_0}$$

  where $\mathbf{L}_0 =$ Moser's $\mathbf{L}|_{t=0}$ and $\mathbf{X}_0 = \mathrm{diag}(x_1(0), \ldots, x_N(0))$ (initial positions).
- **Eigenvalues** $\{x_k(t)\}_{k=1}^N$ of matrix $\mathbf{X}(t)$ **solve CM-system**.
- Matrix approach yields a whole **zoo of CM-systems** with Lie algebras, $\wp(x)$ instead of $1/x^2$, etc.

From discrete to continuum CM-systems

Question: What happens to CM-systems as $N \to \infty$?

From discrete to continuum CM-systems

Question: What happens to CM-systems as $N \to \infty$?

Scalar Continuum CM-Systems:

- **Calogero-Moser Derivative NLS:**

$$\boxed{\mathrm{i}\partial_t u = -\partial_{xx} u + (D + |D|)(|u|^2)u}$$

[Abanov-Bettelheim-Wiegmann '09, L.-Gérard '22, Killip et al. '23, Kim et al. '24, Kim-Kwon '24, Badreddine '23, ...]

- **Benjamin-Ono Equation:**

$$\boxed{\partial_t u = \partial_x(|D|u - u^2)}$$

[Ingimarson-Pego '23]...

From discrete to continuum CM-systems

Question: What happens to CM-systems as $N \to \infty$?

Scalar Continuum CM-Systems:

- **Calogero-Moser Derivative NLS:**

$$\boxed{\mathrm{i}\partial_t u = -\partial_{xx} u + (D + |D|)(|u|^2)u}$$

[Abanov-Bettelheim-Wiegmann '09, L.-Gérard '22, Killip et al. '23, Kim et al. '24, Kim-Kwon '24, Badreddine '23, ...]

- **Benjamin-Ono Equation:**

$$\boxed{\partial_t u = \partial_x(|D|u - u^2)}$$

[Ingimarson-Pego '23]...

Spin Continnum CM-Systems:

- **Half-Wave Maps Equation:**

$$\boxed{\partial_t \mathbf{u} = \mathbf{u} \times |D|\mathbf{u}} \tag{HWM}$$

[Zhou-Stone '17, L-Schikorra '18, L.-Sok '20, Langmann et al. '22, Matsuno '22]...

Common Features of Continuum CM-Systems

- **Lax pair** structure on L^2-based **Hardy spaces**

$$L^2_+ = \{f \in L^2 \mid \operatorname{supp} \widehat{f} \subset [0, \infty)\}$$

Prototypes are rational functions with poles in $\mathbb{C}_-$, e. g., $f(x) = \frac{1}{x+\mathrm{i}} \in L^2_+$

Common Features of Continuum CM-Systems

- **Lax pair** structure on L^2-based **Hardy spaces**

$$L^2_+ = \{f \in L^2 \mid \operatorname{supp} \widehat{f} \subset [0, \infty)\}$$

  Prototypes are rational functions with poles in $\mathbb{C}_-$, e. g., $f(x) = \frac{1}{x+\mathrm{i}} \in L^2_+$

- **Infinite hierarchy of conserved quantities**

$$I_k(u(t)) = I_k(u_0) \quad \text{for } k = 1, 2, 3, \dots$$

- **Solitons** are always **rational functions**

Common Features of Continuum CM-Systems

- **Lax pair** structure on L^2-based **Hardy spaces**

$$L^2_+ = \{f \in L^2 \mid \operatorname{supp} \widehat{f} \subset [0, \infty)\}$$

Prototypes are rational functions with poles in $\mathbb{C}_-$, e. g., $f(x) = \frac{1}{x+\mathrm{i}} \in L^2_+$

- **Infinite hierarchy of conserved quantities**

$$I_k(u(t)) = I_k(u_0) \quad \text{for } k = 1, 2, 3, \ldots$$

- **Solitons** are always **rational functions**
- Existence of **turbulent solutions** with growth of Sobolev norms

$$\|u(t)\|_{H^s} \to +\infty \quad \text{as } t \to +\infty$$

for $s > s_*$. In fact, **finite-time blowup** is known for (CM-DNLS)

Common Features of Continuum CM-Systems

- **Lax pair** structure on L^2-based **Hardy spaces**

$$L^2_+ = \{f \in L^2 \mid \operatorname{supp} \widehat{f} \subset [0, \infty)\}$$

  Prototypes are rational functions with poles in $\mathbb{C}_-$, e. g., $f(x) = \frac{1}{x+\mathrm{i}} \in L^2_+$

- **Infinite hierarchy of conserved quantities**

$$I_k(u(t)) = I_k(u_0) \quad \text{for } k = 1, 2, 3, \ldots$$

- **Solitons** are always **rational functions**
- Existence of **turbulent solutions** with growth of Sobolev norms

$$\|u(t)\|_{H^s} \to +\infty \quad \text{as } t \to +\infty$$

  for $s > s_*$. In fact, **finite-time blowup** is known for (CM-DNLS)

- **Explicit flow formulas** (see below)

Common Features of Continuum CM-Systems

- **Lax pair** structure on L^2-based **Hardy spaces**

$$L^2_+ = \{f \in L^2 \mid \operatorname{supp} \widehat{f} \subset [0,\infty)\}$$

  Prototypes are rational functions with poles in $\mathbb{C}_-$, e. g., $f(x) = \frac{1}{x+\mathrm{i}} \in L^2_+$
- **Infinite hierarchy of conserved quantities**

$$I_k(u(t)) = I_k(u_0) \quad \text{for } k = 1, 2, 3, \ldots$$

- **Solitons** are always **rational functions**
- Existence of **turbulent solutions** with growth of Sobolev norms

$$\|u(t)\|_{H^s} \to +\infty \quad \text{as } t \to +\infty$$

  for $s > s_*$. In fact, **finite-time blowup** is known for (CM-DNLS)
- **Explicit flow formulas** (see below)

For rest of my talk, I focus on (HWM)

Agenda:

1. What are Continuum CM-Systems?
2. Half-Wave Maps Equation as Pathfinder
3. Recent Results
4. Outlook & Open Questions

Introducing the Half-Wave Maps Equation

For maps $\mathbf{u} : [0, T) \times \mathbb{R} \to \mathbb{S}^2$, we consider the evolution equation

$$\boxed{\partial_t \mathbf{u} = \mathbf{u} \times |D|\mathbf{u}} \tag{HWM}$$

Here $\mathbb{S}^2 \subset \mathbb{R}^3$ is unit sphere, $\times$ is cross product, and $\widehat{|D|f} = |\xi|\widehat{f}$.

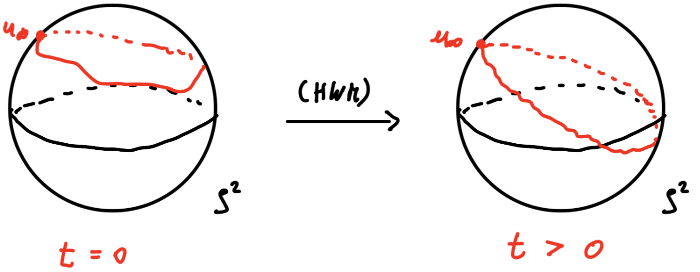
Introducing the Half-Wave Maps Equation

For maps $\mathbf{u} : [0, T) \times \mathbb{R} \to \mathbb{S}^2$, we consider the evolution equation

$$\boxed{\partial_t \mathbf{u} = \mathbf{u} \times |D|\mathbf{u}} \tag{HWM}$$

Here $\mathbb{S}^2 \subset \mathbb{R}^3$ is unit sphere, $\times$ is cross product, and $\widehat{|D|f} = |\xi|\widehat{f}$.

- For smooth initial data $\mathbf{u}_0 = \mathbf{u}_\infty + \mathbf{v} \in \mathbb{S}^2 + H^\infty(\mathbb{R}; \mathbb{R}^3)$, we have short time existence.
- (HWM) as Hamiltonian flow of closed curves on $\mathbb{S}^2$



Introducing the Half-Wave Maps Equation

For maps $\mathbf{u} : [0, T) \times \mathbb{R} \to \mathbb{S}^2$, we consider the evolution equation

$$\boxed{\partial_t \mathbf{u} = \mathbf{u} \times |D|\mathbf{u}} \tag{HWM}$$

Here $\mathbb{S}^2 \subset \mathbb{R}^3$ is unit sphere, $\times$ is cross product, and $\widehat{|D|f} = |\xi|\widehat{f}$.

- Hamiltonian flow for 'half-Dirichlet energy' with

$$E(\mathbf{u}) = \frac{1}{2}\int_{\mathbb{R}} \mathbf{u} \cdot |D|\mathbf{u} = \frac{1}{4\pi}\iint_{\mathbb{R}\times\mathbb{R}} \frac{|\mathbf{u}(x) - \mathbf{u}(y)|^2}{|x-y|^2}\,dx\,dy.$$

  Crit. pts. of $E(\mathbf{u})$ = stationary solutions of (HWM) = **half-harmonic maps**
- Energy-critical because of scaling

$$\mathbf{u}(t,x) \mapsto \mathbf{u}(\lambda t, \lambda x)$$

Introducing the Half-Wave Maps Equation

For maps $\mathbf{u} : [0, T) \times \mathbb{R} \to \mathbb{S}^2$, we consider the evolution equation

$$\boxed{\partial_t \mathbf{u} = \mathbf{u} \times |D|\mathbf{u}} \qquad \text{(HWM)}$$

Here $\mathbb{S}^2 \subset \mathbb{R}^3$ is unit sphere, $\times$ is cross product, and $\widehat{|D|f} = |\xi|\widehat{f}$.

- Hamiltonian flow for 'half-Dirichlet energy' with

$$E(\mathbf{u}) = \frac{1}{2}\int_{\mathbb{R}} \mathbf{u} \cdot |D|\mathbf{u} = \frac{1}{4\pi}\iint_{\mathbb{R}\times\mathbb{R}} \frac{|\mathbf{u}(x) - \mathbf{u}(y)|^2}{|x-y|^2}\, dx\, dy.$$

  Crit. pts. of $E(\mathbf{u})$ = stationary solutions of (HWM) = **half-harmonic maps**
- Energy-critical because of scaling

$$\mathbf{u}(t,x) \mapsto \mathbf{u}(\lambda t, \lambda x)$$

- Completely integrable with Lax pair found bin [L.-Gérard '18].
- Pole dynamics studied in [Zhou-Stone '17], [Langmann et. al '22], [Matsuno '22].

Lax Structure for (HWM)

For smooth solutions $\mathbf{u}(t)$ of (HWM), find **Lax equation** of commutator form

$$\frac{d}{dt} L_{\mathbf{u}(t)} = [B_{\mathbf{u}(t)}, L_{\mathbf{u}(t)}]$$

with (possibly unbounded) operators $L^*_{\mathbf{u}(t)} = L_{\mathbf{u}(t)}$ and $B^*_{\mathbf{u}(t)} = -B_{\mathbf{u}(t)}$ acting on some Hilbert space $\mathcal{H}$.

Lax Structure for (HWM)

For smooth solutions $\mathbf{u}(t)$ of (HWM), find **Lax equation** of commutator form

$$\frac{d}{dt} L_{\mathbf{u}(t)} = [B_{\mathbf{u}(t)}, L_{\mathbf{u}(t)}]$$

with (possibly unbounded) operators $L^*_{\mathbf{u}(t)} = L_{\mathbf{u}(t)}$ and $B^*_{\mathbf{u}(t)} = -B_{\mathbf{u}(t)}$ acting on some Hilbert space $\mathcal{H}$.

- If Lax pair $(L_{\mathbf{u}}, B_{\mathbf{u}})$ exists, we (formally) have

$$L_{\mathbf{u}(t)} = \mathcal{U}(t) L_{\mathbf{u}(0)} \mathcal{U}(t)^*$$

  where $\mathcal{U}(t)$ is unitary map on $\mathcal{H}$ generated by $B_{\mathbf{u}(t)}$.
- Spectrum $\sigma(L_{\mathbf{u}(t)})$ preserved. Infinite hierarchy of conserved quantities with

$$I_m(\mathbf{u}(t)) = \mathrm{Tr}\left(|L_{\mathbf{u}}(t)|^m\right) = \text{const.} \quad \text{for } m = 1, 2, 3, \ldots$$

Lax Structure for (HWM)

For smooth solutions $\mathbf{u}(t)$ of (HWM), find **Lax equation** of commutator form

$$\frac{d}{dt} L_{\mathbf{u}(t)} = [B_{\mathbf{u}(t)}, L_{\mathbf{u}(t)}]$$

with (possibly unbounded) operators $L^*_{\mathbf{u}(t)} = L_{\mathbf{u}(t)}$ and $B^*_{\mathbf{u}(t)} = -B_{\mathbf{u}(t)}$ acting on some Hilbert space $\mathcal{H}$.

- If Lax pair $(L_{\mathbf{u}}, B_{\mathbf{u}})$ exists, we (formally) have

$$L_{\mathbf{u}(t)} = \mathcal{U}(t) L_{\mathbf{u}(0)} \mathcal{U}(t)^*$$

  where $\mathcal{U}(t)$ is unitary map on $\mathcal{H}$ generated by $B_{\mathbf{u}(t)}$.
- Spectrum $\sigma(L_{\mathbf{u}(t)})$ preserved. Infinite hierarchy of conserved quantities with

$$I_m(\mathbf{u}(t)) = \mathrm{Tr}\left(|L_{\mathbf{u}}(t)|^m\right) = \text{const.} \quad \text{for } m = 1, 2, 3, \ldots$$

- **Question:** What are $(L_{\mathbf{u}}, B_{\mathbf{u}}, \mathcal{H})$ for (HWM)?

By algebraic trick, inspired by Takhtajan/Faddeev for (SM), we consider matrix-valued version

$$\partial_t \mathbf{U} = \frac{\mathrm{i}}{2}[\mathbf{U}, |D|\mathbf{U}] \tag{HWM}$$

with the **matrix-valued function U** using the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ where

$$\mathbf{U} = \mathbf{u} \cdot \boldsymbol{\sigma} = \begin{pmatrix} u_3 & u_1 - \mathrm{i}u_2 \\ u_1 + \mathrm{i}u_2 & -u_3 \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

- Note that $\mathbf{U}^* = \mathbf{U}$ and $\mathbf{U}^2 = \mathbb{1}_2$ and $\mathrm{Tr}\,\mathbf{U} = 0$.

By algebraic trick, inspired by Takhtajan/Faddeev for (SM), we consider matrix-valued version

$$\partial_t \mathbf{U} = \frac{\mathrm{i}}{2}[\mathbf{U}, |D|\mathbf{U}] \tag{HWM}$$

with the **matrix-valued function U** using the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ where

$$\mathbf{U} = \mathbf{u} \cdot \boldsymbol{\sigma} = \begin{pmatrix} u_3 & u_1 - \mathrm{i}u_2 \\ u_1 + \mathrm{i}u_2 & -u_3 \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

- Note that $\mathbf{U}^* = \mathbf{U}$ and $\mathbf{U}^2 = \mathbb{1}_2$ and $\mathrm{Tr}\,\mathbf{U} = 0$.
- **Attempt #1:** Let μ_f be multiplication operator with symbol f. Take

$$L_{\mathbf{u}} = \mu_{\mathbf{U}} \quad \text{and} \quad B_{\mathbf{u}} = -\frac{\mathrm{i}}{2}\mu_{|D|\mathbf{U}}$$

  acting on $\mathcal{H} = L^2(\mathbb{R}; \mathbb{C}^2)$.
- Then (HWM) **is** already a Lax equation.

By algebraic trick, inspired by Takhtajan/Faddeev for (SM), we consider matrix-valued version

$$\partial_t \mathbf{U} = \frac{\mathrm{i}}{2}[\mathbf{U}, |D|\mathbf{U}] \tag{HWM}$$

with the **matrix-valued function U** using the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ where

$$\mathbf{U} = \mathbf{u} \cdot \boldsymbol{\sigma} = \begin{pmatrix} u_3 & u_1 - \mathrm{i} u_2 \\ u_1 + \mathrm{i} u_2 & -u_3 \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

- Note that $\mathbf{U}^* = \mathbf{U}$ and $\mathbf{U}^2 = \mathbb{1}_2$ and $\mathrm{Tr}\,\mathbf{U} = 0$.
- **Attempt #1:** Let μ_f be multiplication operator with symbol f. Take

$$L_{\mathbf{u}} = \mu_{\mathbf{U}} \quad \text{and} \quad B_{\mathbf{u}} = -\frac{\mathrm{i}}{2}\mu_{|D|\mathbf{U}}$$

  acting on $\mathcal{H} = L^2(\mathbb{R}; \mathbb{C}^2)$.
- Then (HWM) **is** already a Lax equation.
- But we always have $\sigma(L_{\mathbf{u}}) = \{\pm 1\}$ is trivial!

By algebraic trick, inspired by Takhtajan/Faddeev for (SM), we consider matrix-valued version

$$\partial_t \mathbf{U} = \frac{\mathrm{i}}{2}[\mathbf{U}, |D|\mathbf{U}] \tag{HWM}$$

with the **matrix-valued function U** using the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ where

$$\mathbf{U} = \mathbf{u} \cdot \boldsymbol{\sigma} = \begin{pmatrix} u_3 & u_1 - \mathrm{i}u_2 \\ u_1 + \mathrm{i}u_2 & -u_3 \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

- **Attempt #2:** Using that $\mathbf{U}^2 = \mathbb{1}_2$ we can freely change to

$$B_{\mathbf{u}} = \frac{\mathrm{i}}{2}\left(\mu_{\mathbf{U}} \circ |D| + |D| \circ \mu_{\mathbf{U}}\right) - \frac{\mathrm{i}}{2}\mu_{|D|\mathbf{U}}$$

With this choice, we get a **decent Lax** equation

$$\frac{d}{dt}L_{\mathbf{u}} = [B_{\mathbf{u}}, L_{\mathbf{u}}] \quad \text{for any choice of } L_{\mathbf{u}} \in \{\mu_{\mathbf{U}}, \Pi_+, \Pi_-\}$$

where $\Pi_\pm : L^2 \to L^2_\pm$ are **Cauchy–Szegő projections**, i. e., in Fourier space we have

$$\widehat{(\Pi_+ f)}(\xi) = \mathbb{1}_{\xi \geq 0}\widehat{f}(\xi)$$

Exploiting the Lax Structure for (HWM)

Consider smooth solutions $\mathbf{U}(t)$ to matrix-version of (HWM) with

$$\partial_t \mathbf{U} = \frac{\mathrm{i}}{2}[\mathbf{U}, |D|\mathbf{U}]$$

- By Leibniz rule for $\frac{d}{dt}$ and $[X, Y]$, we see <u>any</u> operator product

$$\mu_{\mathbf{u}}, \Pi_+ \mu_{\mathbf{U}} \Pi_+, \Pi_+ \mu_{\mathbf{U}} \Pi_-, \ldots$$

  is a Lax operator for (HWM) with spectrum preserved in time.

Exploiting the Lax Structure for (HWM)

Consider smooth solutions $\mathbf{U}(t)$ to matrix-version of (HWM) with

$$\partial_t \mathbf{U} = \frac{\mathrm{i}}{2}[\mathbf{U}, |D|\mathbf{U}]$$

- By Leibniz rule for $\frac{d}{dt}$ and $[X, Y]$, we see <u>any</u> operator product

$$\mu_{\mathbf{u}}, \Pi_+ \mu_{\mathbf{U}} \Pi_+, \Pi_+ \mu_{\mathbf{U}} \Pi_-, \ldots$$

  is a Lax operator for (HWM) with spectrum preserved in time.
- Most notably, we have the pair of **Lax operators** given by

$$\boxed{T_{\mathbf{U}} = \Pi_+ \mu_{\mathbf{U}} \Pi_+ \quad \text{and} \quad H_{\mathbf{U}} = \Pi_- \mu_{\mathbf{U}} \Pi_+}$$

  referred to as **Toeplitz** and **Hankel operator** with (matrix-valued) symbol $\mathbf{U}$.

Exploiting the Lax Structure for (HWM)

Consider smooth solutions $\mathbf{U}(t)$ to matrix-version of (HWM) with

$$\partial_t \mathbf{U} = \frac{\mathrm{i}}{2}[\mathbf{U}, |D|\mathbf{U}]$$

- By Leibniz rule for $\frac{d}{dt}$ and $[X, Y]$, we see any operator product

$$\mu_{\mathbf{u}}, \Pi_+ \mu_{\mathbf{U}} \Pi_+, \Pi_+ \mu_{\mathbf{U}} \Pi_-, \ldots$$

  is a Lax operator for (HWM) with spectrum preserved in time.
- Most notably, we have the pair of **Lax operators** given by

$$\boxed{T_{\mathbf{U}} = \Pi_+ \mu_{\mathbf{U}} \Pi_+ \quad \text{and} \quad H_{\mathbf{U}} = \Pi_- \mu_{\mathbf{U}} \Pi_+}$$

  referred to as **Toeplitz** and **Hankel operator** with (matrix-valued) symbol $\mathbf{U}$.
- From $\mathbf{U}^2 = \mathbb{1}_2$ and $\mathbf{U}^* = \mathbf{U}$, we derive **key identity**

$$\boxed{T_{\mathbf{U}}^2 = \mathrm{Id} - K_{\mathbf{U}}}$$

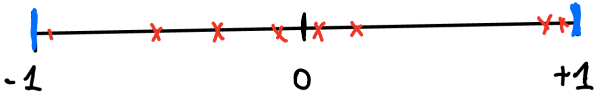
  with trace-class operator $K_{\mathbf{U}} = H_{\mathbf{U}}^* H_{\mathbf{U}}$ with $\mathrm{Tr}(K_{\mathbf{U}}) \sim \|\mathbf{u}\|^2_{\dot{H}^{\frac{1}{2}}}$.

Spectal Analysis of $T_{\mathbf{U}}$

For general data $\mathbf{u} \in \dot{H}^{\frac{1}{2}}(\mathbb{R};\mathbb{S}^2)$, we have

$$\boxed{T_{\mathbf{U}}^2 = \mathrm{Id} - K_{\mathbf{U}} \qquad \text{with } 0 \leq K_{\mathbf{U}} \leq 1 \text{ trace-class}}$$

Spectrum of $T_{\mathbf{U}}$ decompesed as $\sigma(T_{\mathbf{U}}) = \sigma_{\mathrm{d}}(T_{\mathbf{U}}) \cup \sigma_{\mathrm{ess}}(T_{\mathbf{U}})$



- Infinite set of conserved quantities by p-**Schatten norms** of $K_{\mathbf{U}}$

$$\mathrm{Tr}(K_{\mathbf{U}}^p) = \|K_{\mathbf{U}}\|_{\mathfrak{S}^p}^p \sim \|\mathbf{u}\|_{\dot{B}_p^{1/p}}^p \quad \text{(Besov norms)}$$

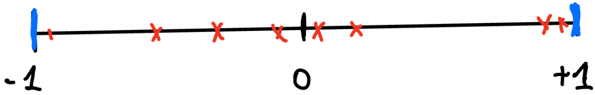
But quantities not strong enough to extend solutions globally in time!

Spectal Analysis of $T_{\mathbf{u}}$

For general data $\mathbf{u} \in \dot{H}^{\frac{1}{2}}(\mathbb{R};\mathbb{S}^2)$, we have

$$\boxed{T_{\mathbf{u}}^2 = \mathrm{Id} - K_{\mathbf{u}} \qquad \text{with } 0 \leq K_{\mathbf{u}} \leq 1 \text{ trace-class}}$$

Spectrum of $T_{\mathbf{u}}$ decompesed as $\sigma(T_{\mathbf{u}}) = \sigma_{\mathrm{d}}(T_{\mathbf{u}}) \cup \sigma_{\mathrm{ess}}(T_{\mathbf{u}})$



- Infinite set of conserved quantities by p-**Schatten norms** of $K_{\mathbf{u}}$

$$\mathrm{Tr}(K_{\mathbf{u}}^p) = \|K_{\mathbf{u}}\|_{\mathfrak{S}^p}^p \sim \|\mathbf{u}\|_{\dot{B}_p^{1/p}}^p \quad \text{(Besov norms)}$$

  But quantities not strong enough to extend solutions globally in time!
- Kronecker's theorem for Hankel operators yields

$$\mathrm{rank}\,(K_{\mathbf{u}}) < \infty \quad \Leftrightarrow \quad \mathbf{u} \text{ is rational}$$

  Thus rationality is preserevd by flow of (HWM)!

Agenda:

1. What are Continuum CM-Systems?
2. Half-Wave Maps Equation as Pathfinder
3. Recent Results
4. Outlook & Open Questions

GLobal Wellposedness for Rational Data

Theorem (L.-Gérard '24)

(HWM) is globally wellposed for rational initial data, i. e., for any $\mathbf{u}_0 \in \mathcal{R}at(\mathbb{R}; \mathbb{S}^2)$ *there exists a unique solution* $\mathbf{u} \in C(\mathbb{R}; \dot{H}^\infty)$ *of (HWM) with* $\mathbf{u}(0) = \mathbf{u}_0$*.*

Remarks

- **First GWP-result** for (HWM). Also **no small data** result.
- In fact, the set $\mathcal{R}at(\mathbb{R}; \mathbb{S}^2)$ is **dense** in the energy space $\dot{H}^{\frac{1}{2}}(\mathbb{R}; \mathbb{S}^2)$.

GLobal Wellposedness for Rational Data

Theorem (L.-Gérard '24)

(HWM) is globally wellposed for rational initial data, i. e., for any $\mathbf{u}_0 \in \mathcal{R}at(\mathbb{R}; \mathbb{S}^2)$ *there exists a unique solution* $\mathbf{u} \in C(\mathbb{R}; \dot{H}^\infty)$ *of (HWM) with* $\mathbf{u}(0) = \mathbf{u}_0$.

Remarks

- **First GWP-result** for (HWM). Also **no small data** result.
- In fact, the set $\mathcal{R}at(\mathbb{R}; \mathbb{S}^2)$ is **dense** in the energy space $\dot{H}^{\frac{1}{2}}(\mathbb{R}; \mathbb{S}^2)$.
- Proof uses explicit flow formula for (HWM), valid for sufficiently smooth solutions of (HWM) which are not necessarily rational.
- However, our analysis becomes feasible for rational solutions.

Explicit Flow Formula

Let $\mathbf{U}(t) = \mathbf{U}_\infty + \mathbf{V}(t)$ be smooth solution of initial-value problem

$$\partial_t \mathbf{U} = \frac{\mathrm{i}}{2}[\mathbf{U}, |D|\mathbf{U}] \quad \text{for } t \in [0, T], \quad \mathbf{U}(0) = \mathbf{U}_0 \,.$$

Because of $\mathbf{U} = \mathbf{U}^*$, all information is contained in the part $\Pi_+ \mathbf{V}(t) \in L^2_+$.

Explicit Flow Formula

Let $\mathbf{U}(t) = \mathbf{U}_\infty + \mathbf{V}(t)$ be smooth solution of initial-value problem

$$\partial_t \mathbf{U} = \frac{\mathrm{i}}{2}[\mathbf{U}, |D|\mathbf{U}] \quad \text{for } t \in [0, T], \quad \mathbf{U}(0) = \mathbf{U}_0 \,.$$

Because of $\mathbf{U} = \mathbf{U}^*$, all information is contained in the part $\Pi_+ \mathbf{V}(t) \in L^2_+$.

Lemma (Explicit Flow Formula)

For all $t \in [0, T]$, it holds

$$\boxed{\Pi_+ \mathbf{V}(t, z) = \frac{1}{2\pi \mathrm{i}} I_+ \left[(X^* + t T_{\mathbf{U}_0} - z)^{-1} \Pi_+ \mathbf{V}_0 \right] \quad \forall\, z \in \mathbb{C}_+}$$

- Here Toeplitz operator $T_{\mathbf{U}_0}$ acts on $L^2_+(\mathbb{R}; \mathbb{C}^{2\times 2})$ and **not** on $L^2_+(\mathbb{R}; \mathbb{C}^2)$
- $(A - z)^{-1}$ resolvent of operator A with $z \in \rho(A)$.
- For definition of operators X^* and I_+, see next slide.

Explicit Flow Formula

Let $\mathbf{U}(t) = \mathbf{U}_\infty + \mathbf{V}(t)$ be smooth solution of initial-value problem

$$\partial_t \mathbf{U} = \frac{\mathrm{i}}{2}[\mathbf{U}, |D|\mathbf{U}] \quad \text{for } t \in [0, T], \quad \mathbf{U}(0) = \mathbf{U}_0 \,.$$

Because of $\mathbf{U} = \mathbf{U}^*$, all information is contained in the part $\Pi_+ \mathbf{V}(t) \in L^2_+$.

Lemma (Explicit Flow Formula)

For all $t \in [0, T]$, it holds

$$\boxed{\Pi_+ \mathbf{V}(t, z) = \frac{1}{2\pi \mathrm{i}} I_+ \left[(X^* + t T_{\mathbf{U}_0} - z)^{-1} \Pi_+ \mathbf{V}_0 \right] \quad \forall\, z \in \mathbb{C}_+}$$

- Here Toeplitz operator $T_{\mathbf{U}_0}$ acts on $L^2_+(\mathbb{R}; \mathbb{C}^{2\times 2})$ and **not** on $L^2_+(\mathbb{R}; \mathbb{C}^2)$
- $(A - z)^{-1}$ resolvent of operator A with $z \in \rho(A)$.
- For definition of operators X^* and I_+, see next slide.
- RHS is well-defined for **all** $t \in \mathbb{R}$. So why not GWP? Things are delicate!

Constuction of X^*

X^* is renormalized version of position operator $x \mapsto xf$ on L^2_+

- The operator $X^* : \mathrm{dom}(X^*) \subset L^2_+ \to L^2_+$ is defined as

$$\widehat{(X^* f)}(\xi) := \mathrm{i}\frac{d\widehat{f}}{d\xi} \quad \text{with} \quad \mathrm{dom}(X^*) = \{f \in L^2_+ \mid \frac{d\widehat{f}}{d\xi} \in L^2_+\}$$

Constuction of X^*

X^* is renormalized version of position operator $x \mapsto xf$ on L^2_+

- The operator $X^* : \mathrm{dom}(X^*) \subset L^2_+ \to L^2_+$ is defined as

$$\widehat{(X^* f)}(\xi) := \mathrm{i}\frac{d\widehat{f}}{d\xi} \quad \text{with} \quad \mathrm{dom}(X^*) = \{f \in L^2_+ \mid \frac{d\widehat{f}}{d\xi} \in L^2_+\}$$

- Its adjoint $X \neq X^*$ has domain $\mathrm{dom}(X) = \{f \in \mathrm{dom}(X^*) \mid \widehat{f}(0^+) = 0\}$
- **Rational functions** in L^2_+ belong to $\mathrm{dom}(X^*)$ but **not** to $\mathrm{dom}(X)$, e. g.,

$$\frac{1}{x+\mathrm{i}} \in \mathrm{dom}(X^*) \quad \text{but} \quad \frac{1}{x+\mathrm{i}} \notin \mathrm{dom}(X)\,.$$

Constuction of X^*

X^* is renormalized version of position operator $x \mapsto xf$ on L^2_+

- The operator $X^* : \mathrm{dom}(X^*) \subset L^2_+ \to L^2_+$ is defined as

$$\widehat{(X^* f)}(\xi) := \mathrm{i}\frac{d\widehat{f}}{d\xi} \quad \text{with} \quad \mathrm{dom}(X^*) = \{f \in L^2_+ \mid \frac{d\widehat{f}}{d\xi} \in L^2_+\}$$

- Its adjoint $X \neq X^*$ has domain $\mathrm{dom}(X) = \{f \in \mathrm{dom}(X^*) \mid \widehat{f}(0^+) = 0\}$
- **Rational functions** in L^2_+ belong to $\mathrm{dom}(X^*)$ but **not** to $\mathrm{dom}(X)$, e. g.,

$$\frac{1}{x+\mathrm{i}} \in \mathrm{dom}(X^*) \quad \text{but} \quad \frac{1}{x+\mathrm{i}} \notin \mathrm{dom}(X)\,.$$

- X and X^* are **generators** of canonical **semigroups** on L^2_+, i. e.

$$\{S(\eta)\}_{\eta \geq 0} = e^{\mathrm{i}\eta X} \ \text{(right shifts)}, \quad \{S^*(\eta)\}_{\eta \geq 0} = e^{\mathrm{i}\eta X^*} \ \text{(left shifts)}\,.$$

Derivation of Explicit Flow Formula

- At time $t = 0$, we have **Cauchy's integral formula** for $f \in L^2_+$ written as

$$\boxed{f(z) = \frac{1}{2\pi \mathrm{i}} I_+[(X^* - z)^{-1} f] \quad \forall\, z \in \mathbb{C}_+}$$

with $I_+(f) := \lim_{\xi \to 0^+} \widehat{f}(\xi)$.

Derivation of Explicit Flow Formula

- At time $t = 0$, we have **Cauchy's integral formula** for $f \in L^2_+$ written as

$$\boxed{f(z) = \frac{1}{2\pi \mathrm{i}} I_+[(X^* - z)^{-1} f] \quad \forall z \in \mathbb{C}_+}$$

  with $I_+(f) := \lim_{\xi \to 0^+} \widehat{f}(\xi)$.
- By **Lax evolution** we get **unitary** map $\mathcal{U}(t)$ and $\Pi_+ \mathbf{V}(t) = \mathcal{U}(t) \Pi \mathbf{V}_0$. By mimicking **Heisenberg's picture** in QM, we get

$$\begin{aligned}
\Pi_+ \mathbf{V}(t, z) &= \frac{1}{2\pi \mathrm{i}} I_+[(X^* - z)^{-1} \mathcal{U}(t) \Pi_+ \mathbf{V}_0] \\
&= \frac{1}{2\pi \mathrm{i}} I_+[(\mathcal{U}(t)^* X^* \mathcal{U}(t) - z)^{-1} \Pi_+ \mathbf{V}_0]
\end{aligned}$$

Derivation of Explicit Flow Formula

- At time $t = 0$, we have **Cauchy's integral formula** for $f \in L^2_+$ written as

$$\boxed{f(z) = \frac{1}{2\pi \mathrm{i}} I_+[(X^* - z)^{-1} f] \quad \forall\, z \in \mathbb{C}_+}$$

  with $I_+(f) := \lim_{\xi \to 0^+} \widehat{f}(\xi)$.

- By **Lax evolution** we get **unitary** map $\mathcal{U}(t)$ and $\Pi_+ \mathbf{V}(t) = \mathcal{U}(t) \Pi \mathbf{V}_0$. By mimicking **Heisenberg's picture** in QM, we get

$$\begin{aligned}
\Pi_+ \mathbf{V}(t, z) &= \frac{1}{2\pi \mathrm{i}} I_+[(X^* - z)^{-1} \mathcal{U}(t) \Pi_+ \mathbf{V}_0] \\
&= \frac{1}{2\pi \mathrm{i}} I_+[(\mathcal{U}(t)^* X^* \mathcal{U}(t) - z)^{-1} \Pi_+ \mathbf{V}_0]
\end{aligned}$$

- Finally, by Lax pair structure and commutator relations,

$$\boxed{\mathcal{U}(t)^* X^* \mathcal{U}(t) = X^* + t T_{\mathbf{U}_0}}$$

- Ersatz of Olshanetsky/Perelomov in infinite dimensions!

Fine structure of resolvent is important to obtain (GWP)!

Fine structure of resolvent is important to obtain (GWP)!

- For **half-wave maps**, we find

$$\Pi_+ \mathbf{V}(t,z) = \frac{1}{2\pi \mathrm{i}} I_+[(X^* + tT_{\mathbf{U}_0} - z)^{-1} \Pi_+ \mathbf{V}_0]$$

  for matrix-valued initial data $\mathbf{U}_0$ with and $\mathbf{U}_0 = \mathbf{U}_0^*$ and $\mathbf{U}_0^2 = \mathbb{1}_2$.

- For **zero-dispersion limit (BO)**, we find

$$\Pi_+ u(t,z) = \frac{1}{2\pi \mathrm{i}} I_+((X^* + tT_{u_0} - z)^{-1} \Pi_+ u_0]$$

  with scalar initial data $u_0 = u_0^*$.
  Singularity formation (shocks!) in finite time is known!

Soliton Resolution and Non-Turbulence

Theorem (L.-Gérard '24)

Let $\mathbf{u}_0 \in \mathcal{R}at(\mathbb{R};\mathbb{S}^2)$ *and suppose that* $T_{\mathbf{U}_0}$ *has simple discrete spectrum*

$$\sigma_{\mathrm{d}}(T_{\mathbf{U}_0}) = \{v_1, \ldots, v_N\}\,.$$

Then corresponding solution $\mathbf{u} \in C(\mathbb{R}; \dot{H}^\infty)$ *exhibits soliton resolution with*

$$\mathbf{u}(t,x) = \sum_{j=1}^{N} \mathbf{q}_{v_j}(x - v_j t) - (N-1)\mathbf{u}_\infty + o_{\dot{H}^s}(1) \quad \textit{as } t \to \pm\infty$$

where each $\mathbf{q}_{v_j}$ *is a ground state traveling solitary wave* profile for *(HWM).*

Soliton Resolution and Non-Turbulence

Theorem (L.-Gérard '24)

Let $\mathbf{u}_0 \in \mathcal{R}at(\mathbb{R}; \mathbb{S}^2)$ *and suppose that* $T_{\mathbf{U}_0}$ *has simple discrete spectrum*

$$\sigma_{\mathrm{d}}(T_{\mathbf{U}_0}) = \{v_1, \ldots, v_N\}.$$

Then corresponding solution $\mathbf{u} \in C(\mathbb{R}; \dot{H}^\infty)$ *exhibits soliton resolution with*

$$\mathbf{u}(t,x) = \sum_{j=1}^{N} \mathbf{q}_{v_j}(x - v_j t) - (N-1)\mathbf{u}_\infty + o_{\dot{H}^s}(1) \quad \text{as } t \to \pm\infty$$

where each $\mathbf{q}_{v_j}$ *is a ground state traveling solitary wave profile for (HWM).*

- Obtain a-priori bounds $\sup_{t \in \mathbb{R}} \|\mathbf{u}(t)\|_{\dot{H}^s} \leq C(\mathbf{u}_0, s)$. No turbulence!
- Set of initial data $\{\mathbf{u}_0 \mid \sigma_{\mathrm{d}}(T_{\mathbf{U}_0}) \text{ simple}\}$ is **dense** in energy space $\dot{H}^{\frac{1}{2}}(\mathbb{R}; \mathbb{S}^2)$.

Soliton Resolution and Non-Turbulence

Theorem (L.-Gérard '24)

Let $\mathbf{u}_0 \in \mathcal{R}at(\mathbb{R}; \mathbb{S}^2)$ *and suppose that* $T_{\mathbf{U}_0}$ *has simple discrete spectrum*

$$\sigma_{\mathrm{d}}(T_{\mathbf{U}_0}) = \{v_1, \ldots, v_N\}\,.$$

Then corresponding solution $\mathbf{u} \in C(\mathbb{R}; \dot{H}^\infty)$ *exhibits soliton resolution with*

$$\mathbf{u}(t,x) = \sum_{j=1}^{N} \mathbf{q}_{v_j}(x - v_j t) - (N-1)\mathbf{u}_\infty + o_{\dot{H}^s}(1) \quad \text{as } t \to \pm\infty$$

where each $\mathbf{q}_{v_j}$ *is a ground state traveling solitary wave profile for (HWM).*

- Obtain a-priori bounds $\sup_{t \in \mathbb{R}} \|\mathbf{u}(t)\|_{\dot{H}^s} \leq C(\mathbf{u}_0, s)$. No turbulence!
- Set of initial data $\{\mathbf{u}_0 \mid \sigma_{\mathrm{d}}(T_{\mathbf{U}_0}) \text{ simple}\}$ is **dense** in energy space $\dot{H}^{\frac{1}{2}}(\mathbb{R}; \mathbb{S}^2)$.
- **Triviality of scattering map** since profiles for $t \to -\infty$ and $t \to +\infty$ conicide.
- Proof uses explicit flow formula.

Soliton Resolution: Idea of Proof

- Let $\varepsilon = t^{-1}$ and write explicit flow formula

$$\Pi_+ \mathbf{V}(\varepsilon, x) = \frac{\varepsilon}{2\pi \mathrm{i}} I_+ \left[(\varepsilon X^* + T_{\mathbf{U}_0} - \varepsilon x)^{-1} \Pi_+ \mathbf{V}_0 \right]$$

- Make perturbative analysis of operator

$$M(\varepsilon) = \varepsilon X^* + T_{\mathbf{U}_0} \quad \text{as } \varepsilon \to 0$$

Soliton Resolution: Idea of Proof

- Let $\varepsilon = t^{-1}$ and write explicit flow formula

$$\Pi_+ \mathbf{V}(\varepsilon, x) = \frac{\varepsilon}{2\pi \mathrm{i}} I_+ \left[(\varepsilon X^* + T_{\mathbf{U}_0} - \varepsilon x)^{-1} \Pi_+ \mathbf{V}_0 \right]$$

- Make perturbative analysis of operator

$$M(\varepsilon) = \varepsilon X^* + T_{\mathbf{U}_0} \quad \text{as } \varepsilon \to 0$$

- For rational data $\mathbf{U}_0$, we only need to consider the restriction

$$M(\varepsilon) = \varepsilon X^* + T_{\mathbf{U}_0} \text{ acting on some subspace } \mathfrak{H} \text{ with } \dim \mathfrak{H} < \infty$$

- Use classical (non-degenerate) perturbation theory for matrices

Soliton Resolution: Idea of Proof

- Let $\varepsilon = t^{-1}$ and write explicit flow formula

$$\Pi_+ \mathbf{V}(\varepsilon, x) = \frac{\varepsilon}{2\pi \mathrm{i}} I_+ \left[(\varepsilon X^* + T_{\mathbf{U}_0} - \varepsilon x)^{-1} \Pi_+ \mathbf{V}_0 \right]$$

- Make perturbative analysis of operator

$$M(\varepsilon) = \varepsilon X^* + T_{\mathbf{U}_0} \quad \text{as } \varepsilon \to 0$$

- For rational data $\mathbf{U}_0$, we only need to consider the restriction

$$M(\varepsilon) = \varepsilon X^* + T_{\mathbf{U}_0} \text{ acting on some subspace } \mathfrak{H} \text{ with } \dim \mathfrak{H} < \infty$$

- Use classical (non-degenerate) perturbation theory for matrices
- Reinstalling time $t = \varepsilon^{-1}$ for $|\varepsilon| \ll 1$, we find resolution

$$\Pi_+ \mathbf{V}(t, x) = \sum_{n=1}^{N} \frac{A_n(t)}{x - z_n(t)} \quad \text{for } |t| \gg 1$$

Refined analysis yields soliton resolution. QED

Outlook and Open Questions

- For (HWM), natural generalization of target $\mathbb{S}^2$ are

$$\mathbf{Gr}_k(\mathbb{C}^d) = \text{complex Grassmannians}$$

Note that $\mathbf{Gr}_1(\mathbb{C}^d) \cong \mathbb{CP}^{d-1}$ and $\mathbb{CP}^1 \cong \mathbb{S}^2$.

Outlook and Open Questions

- For (HWM), natural generalization of target $\mathbb{S}^2$ are

$$\mathbf{Gr}_k(\mathbb{C}^d) = \text{complex Grassmannians}$$

  Note that $\mathbf{Gr}_1(\mathbb{C}^d) \cong \mathbb{CP}^{d-1}$ and $\mathbb{CP}^1 \cong \mathbb{S}^2$.
- Global Wellposedness for non-rational data, turbulence, blow-up?

Outlook and Open Questions

- For (HWM), natural generalization of target $\mathbb{S}^2$ are

$$\mathbf{Gr}_k(\mathbb{C}^d) = \text{complex Grassmannians}$$

  Note that $\mathbf{Gr}_1(\mathbb{C}^d) \cong \mathbb{CP}^{d-1}$ and $\mathbb{CP}^1 \cong \mathbb{S}^2$.
- Global Wellposedness for non-rational data, turbulence, blow-up?
- Systematic study of fine properties of resolvents

$$(X^* + tL_0 - z)^{-1}$$

  for (BO), (CM-DNLS), and (HWM) with operators Lax L_0. Presumably, this is key to GWP and Soliton Resolution...

Thank you for your attention!