Nondegeneracy, stability and symmetry for the fractional Caffarelli–Kohn–Nirenberg inequality

joint work with Nicola De Nitti (EPF Lausanne), Federico Glaudo (Princeton),
arXiv:2403.02303

Gauss Mini Workshop on Analysis, PDEs and Mathematical Physics
TU Braunschweig

Tobias König

Goethe–Universität Frankfurt

January 23, 2025

The classical CKN inequality

Let $n \geq 3$. The **Caffarelli–Kohn–Nirenberg** (CKN) inequality states that

$$\int_{\mathbb{R}^n} |x|^{-2\alpha} |\nabla u|^2 \,\mathrm{d}x \geq \Lambda_{n,\alpha,\beta} \left(\int_{\mathbb{R}^n} |x|^{-\beta p} |u|^p \,\mathrm{d}x \right)^{\frac{2}{p}}, \qquad \text{(CKN)}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$, where $-\infty < \alpha < \frac{n-2}{2} =: \alpha_c$, and $\alpha \leq \beta \leq \alpha + 1$.

- The value of $p = \frac{2n}{n-2+2(\beta-\alpha)}$ is determined by scaling.
- Particular cases include the Hardy ($\alpha = 0$, $\beta = 1$) and Sobolev ($\alpha = \beta = 0$) inequalities.
- (CKN) is invariant under rotation, scaling and inversion.

If $\alpha < \beta < \alpha + 1$, the functions

$$U_{\alpha,\beta}(x) = \left(1 + |x|^{(p-2)(\alpha_c - \alpha)}\right)^{-\frac{2}{p-2}}$$

minimize (CKN) **among radial functions** [Aubin/Talenti 1970s, Chou–Chu 1993, Catrina–Wang 2001]

Symmetry breaking of minimizers

However, there are certain parameter values (α, β) for which **symmetry breaking** occurs! That is, the (global) minimizer (CKN) is a non-radial function.

- There is a curve $(-\infty, 0) \ni \alpha \mapsto \beta_{FS}(\alpha)$ such that for $\beta < \beta_{FS}(\alpha)$ the Hessian of the CKN quotient in $U_{\alpha,\beta}$ has a negative eigenvalue [Felli-Schneider 2003].
- For all $\beta > \beta_{FS}(\alpha)$, all minimizers of (CKN) are radial (and hence given by $U_{\alpha,\beta}$) [Dolbeault–Esteban–Loss 2016]

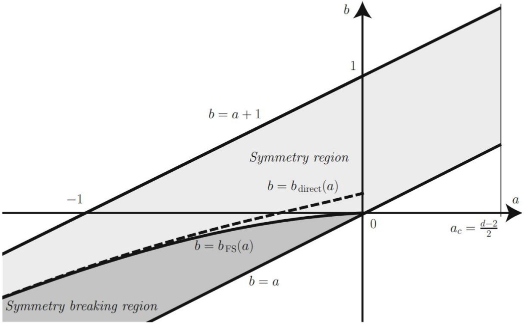

Figure: Symmetry(-breaking) regions for (CKN). Graphic taken from [DEL 2016].

Non-degeneracy and stability

The functions $U_{\alpha,\beta}$ satisfy, for **every** α, β,

$$-\operatorname{div}(|x|^{-2\alpha}\nabla u) = c|x|^{-\beta p}|u|^{p-2}u \qquad \text{on } \mathbb{R}^n. \tag{E}$$

Since $U_\lambda = \lambda^{\frac{n-2\alpha-2}{2}} U(\lambda x)$ solves (E) for all $\lambda > 0$, the derivative $\varphi = \partial_\lambda|_{\lambda=1} U_\lambda$ solves the **linearized equation**

$$-\operatorname{div}(|x|^{-2\alpha}\nabla \varphi) = c(p-1)|x|^{-\beta p}|U|^{p-2}\varphi. \tag{L}$$

- If $\alpha \geq 0$ or if $\beta > \beta_{FS}(\alpha)$, then every solution to (L) is a scalar multiple of $\varphi = \partial_\lambda|_{\lambda=1} U_\lambda$. We say that U is **non-degenerate**.
- If $\alpha < 0$ and $\beta = \beta_{FS}(\alpha)$, then $U_{\alpha,\beta_{FS}(\alpha)}$ is **degenerate**: There is $\varphi \neq \partial_\lambda|_{\lambda=1} U_\lambda$ which solves (L).

In the case where U is a non-degenerate minimizer, one can deduce the **stability** inequality [Bianchi–Egnell 1991, Wei–Wu 2021]

$$\int_{\mathbb{R}^n} |x|^{-2\alpha}|\nabla u|^2 - \Lambda_{n,\alpha,\beta}\left(\int_{\mathbb{R}^n} |x|^{-\beta p}|u|^p\right)^{\frac{2}{p}} \gtrsim \inf_{c\in\mathbb{R},\lambda>0} \int_{\mathbb{R}^n} |x|^{-2\alpha}|\nabla(u - cU_\lambda)|^2$$

For $\beta = \beta_{FS}(\alpha)$, degenerate (quartic) stability still holds [Frank–Peteranderl 2024].

The fractional CKN inequality

Let $n \geq 1$ and $s \in (0, \min\left\{1, \frac{n}{2}\right\})$. A natural counterpart of (CKN) is

$$\|u\|^2_{D^s_\alpha(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x|^\alpha |x-y|^{n+2s} |y|^\alpha} \,\mathrm{d}x \,\mathrm{d}y \geq \Lambda_{n,s,\alpha,\beta} \|u|\cdot|^{-\beta}\|^2_{L^p}, \quad \text{(fCKN)}$$

for $-2s < \alpha < \frac{n-2s}{2}$ and $\alpha \leq \beta \leq \alpha + s$. Again, $p \in [2, \frac{2n}{n-2s}]$ is determined by scaling.

The investigation of (fCKN) in [Ao–DelaTorre–González 2022] has shown:

- A minimizer exists for all $\alpha < \beta < \alpha + s$; moreover, if $0 \leq \alpha < \frac{n-2s}{2}$, it is radially symmetric (see also [Ghoussoub–Shakerian 2015]).
- Minimizers are non-radial for certain choices of the parameters s, p, α, β.
- If the global minimizer is a radial function, then it is non-degenerate **in the space of radial functions** (we will come back to that).
- If the global minimizer is a radial function, then it is unique up to scaling (compare [Frank–Lenzmann 2013, Frank–Lenzmann–Silvestre 2016]).

Difficulties and open questions

The analysis of (fCKN) presents a number of difficulties compared to the classical case:

- The radial minimizers are not explicitly known.
- In particular, there is no explicitly solvable eigenvalue problem leading to an explicit Felli–Schneider curve.
- Transforming to the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ as in [CW 2001, FS 2003, DEL 2016] is still possible, but yields a much more complicated problem than for $s = 1$; see [ADG 2022].

Our new results address the following issues left open by [ADG 2022]:

- **non-degeneracy** (in the full space) of radial minimizers
- sharp quadratic **stability** of (fCKN)
- **symmetry** of minimizers for $-2s < \alpha < 0$.

Differently from [ADG 2022], we work entirely on $\mathbb{R}^n$.

Main result I: Non-degeneracy

The radial minimizer $U \geq 0$ satisfies (after renormalization)

$$\mathcal{L}_{s,\alpha} U(x) := \mathrm{P.V.}_x \int_{\mathbb{R}^n} \frac{U(x) - U(y)}{|x|^\alpha |x-y|^{n+2s} |y|^\alpha} \, \mathrm{d}y = \frac{U^{p-1}}{|x|^{\beta p}}, \tag{fE}$$

and so do $U_\lambda(x) = \lambda^{\frac{n-2s-2\alpha}{2}} U(\lambda x)$, for all $\lambda > 0$. Thus $\varphi = \partial_\lambda|_{\lambda=1} U_\lambda$ solves

$$\mathcal{L}_{s,\alpha} \varphi = (p-1) \frac{U^{p-2}}{|x|^{\beta p}} \varphi. \tag{fL}$$

Theorem (De Nitti, Glaudo, K., 2024)

Let $\alpha \geq 0$. *Let* $0 \leq U \in D^s_\alpha(\mathbb{R}^n)$ *be a solution to* (fE). *Then every solution* $\varphi \in D^s_\alpha(\mathbb{R}^n)$ *to* (fL) *is a scalar multiple of* $\partial_\lambda|_{\lambda=1} U_\lambda$.

- [Musina–Nazarov 2020] prove this for $\alpha = 0$. We use an abstraction of their idea and avoid the Caffarelli–Silvestre extension.

Main result II: Stability

Theorem (De Nitti, Glaudo, K., 2024)

Let $\alpha \geq 0$, and let $U \in D^s_\alpha(\mathbb{R}^n)$ be a minimizer of (fCKN)*. There exists $\kappa > 0$ such that, for all $u \in D^s_\alpha(\mathbb{R}^n)$, it holds*

$$\|u\|^2_{D^s_\alpha(\mathbb{R}^n)} - \Lambda_{n,s,\alpha,\beta} \|u| \cdot |^{-\beta}\|^2_{L^p} \geq \kappa \inf_{c \in \mathbb{R}, \lambda > 0} \|u - cU_\lambda\|^2_{D^s_\alpha(\mathbb{R}^n)},$$

- The key ingredient in the proof is the fact that the linearized operator $\mathcal{L}_{s,\alpha} - (p-1)\frac{U^{p-2}}{|x|^{\beta p}}$ is strictly positive on functions orthogonal to U and $\partial_\lambda U$. This follows from nondegeneracy.
- As a corollary, for all $u \in D^s_\alpha(\mathbb{R}^n)$ supported in some $\Omega \subset \mathbb{R}^n$ we obtain the weak-L^p remainder term estimate

$$\|u\|^2_{D^s_\alpha(\mathbb{R}^n)} - \Lambda_{n,s,\alpha,\beta} \|u| \cdot |^{-\beta}\|^2_{L^p(\Omega)} \geq c|\Omega|^{-\frac{n-2s-2\alpha}{n}} \|u| \cdot |^{-\alpha}\|^2_{L^{\frac{n}{n-2s-\alpha},\infty}}.$$

in the spirit of [Brezis–Lieb 1985, Bianchi–Egnell 1991, Chen–Frank–Weth 2013].

Main result III: Symmetry

Theorem (De Nitti, Glaudo, K., 2024)

For every $-2s < \alpha_0 < 0$, there exists $\varepsilon = \varepsilon(n, s, \alpha_0) > 0$ such that the following statement holds. If $\alpha \in (\alpha_0, 0)$ and $p \in (2, 2 + \varepsilon)$, then every minimizer of (fCKN) *is a radial function.*

- This is the first positive symmetry result for (fCKN) in the region $\alpha < 0$.
- The proof is by contradiction and does not produce an explicit estimate on ε.
- Our proof adapts and generalizes an argument for $s = 1$ from [Dolbeault–Esteban–Loss–Tarantello 2009]. Again, the crucial difficulty to be overcome is the fact that radial minimizers are not explicit. Our main new contribution is the L^∞ bound on minimizers $U_{\alpha,\beta}$

$$\| U_{\alpha,\beta} | \cdot |^{\frac{n-2s-2\alpha}{2}} \|_{L^\infty} \leq M \| U_{\alpha,\beta} | \cdot |^{-\beta} \|_{L^p}^{p-1} + \| U_{\alpha,\beta} | \cdot |^{-\beta} \|_{L^p}^{(p-1)^\kappa}$$

for some M, κ, uniformly in $\alpha \in (\alpha_0, 0)$ and β away from α.

A useful reformulation of (fCKN)

Setting $w(x) = |x|^{-\alpha} u(x)$, (fCKN) becomes equivalently

$$\|w\|_{\dot{H}^s}^2 + C(\alpha) \|w| \cdot|^{-2s}\|_{L^2} \geq \Lambda_{n,s,\alpha,p} \|w| \cdot |^{-t}\|_{L^p}. \qquad \text{(fCKN-H)}$$

for $-2s < \alpha < \frac{n-2s}{2}$ and $p \in (2, \frac{2n}{n-2s})$. The number $t = s - n\left(\frac{1}{2} - \frac{1}{p}\right)$ is determined by scaling. Here,

$$\|w\|_{\dot{H}^s}^2 = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \, \mathrm{d}x \mathrm{d}y$$

and

$$C(\alpha) = C_{n,s} \mathrm{P.V.}_{e_1} \int_{\mathbb{R}^n} \frac{1 - |z|^\alpha}{|z|^\alpha |e_1 - z|^{n+2s}} \, \mathrm{d}z \qquad \in (-C_{\mathsf{Hardy}}(s), +\infty).$$

- Advantages: standard $\dot{H}^s$-norm, rearrangement, angular momentum decomposition.
- $C(\alpha) \leq 0$ iff $\alpha \geq 0$. This causes the restriction $\alpha \geq 0$ in our theorems.

A general Hardy-type inequality

Proposition (De Nitti, Glaudo, K., 2024)

Let U radial with $U' < 0$. For any $\varphi \in \dot{H}^s(\mathbb{R}^n)$ such that $\int_{B_R} \varphi = 0$ for all $R > 0$, we have

$$\|\varphi\|_{\dot{H}^s}^2 \geq \int_{\mathbb{R}^n} \varphi^2 \rho_U, \qquad \text{where } \rho_U := \frac{((-\Delta)^s U)'}{U'}. \tag{1}$$

- [Frank–Seiringer 2008] prove (1) for $\tilde{\rho}_U = \frac{(-\Delta)^s U}{U}$ with $U > 0$ (without condition on φ).
- $U = |x|^{-\alpha}$ gives $\tilde{\rho}_U = c_\alpha |x|^{-2s}$ and $\rho_U = \frac{\alpha+2s}{\alpha} c_\alpha |x|^{-2s}$. This recovers the Hardy inequality, respectively its improvement for φ orthogonal to radial functions [Yafaev 1994].
- The orthogonality requirement on φ is necessary. Indeed, if $(-\Delta)^s U = U^q$ with $q = \frac{n+2s}{n-2s}$, then $\rho_U = qU^{q-1}$. Then $\varphi = U$ gives $\int_{\mathbb{R}^n} \varphi^2 \rho_U = q\|\varphi\|_{\dot{H}^s}^2$.
- Independently, an inequality similar in spirit has been applied in [Fall–Weth 2023] to prove non-degeneracy for a fractional NLS equation.
- The statement also holds for $s = 1$ with a simpler proof.

Proof of non-degeneracy

Let $f(r,z) := r^{-tp} z^{p-1} - C(\alpha) r^{-2s} z$. Then the versions of (fE) and (fL) for a minimizer $W \geq 0$ of (fCKN-H) read

$$(-\Delta)^s W = f(|x|, W), \qquad \text{(fE-H)}$$

$$(-\Delta)^s \varphi = \partial_2 f(|x|, W)\varphi. \qquad \text{(fL-H)}$$

Let φ solve (fL-H), and decompose

$$\varphi = \varphi_0 + \tilde{\varphi}, \qquad \text{such that } \varphi_0 \text{ radial and } \int_{B_r} \tilde{\varphi} = 0 \text{ for all } r > 0.$$

Since $(-\Delta)^s$ and multiplication with $\partial_2 f(|x|, W)$ preserve the space of radial functions and its orthogonal, both φ_0 and $\tilde{\varphi}$ solve (fL-H).

- By the radial non-degeneracy result from [ADM 2022], $\varphi_0 = c \partial_\lambda|_{\lambda=1} W_\lambda$.
- Since $C(\alpha) \leq 0$, we have $\partial_1 f(r, W) < 0$ and so

  $$((-\Delta)^s W)' = \partial_1 f(|x|, W) + \partial_2 f(|x|, W) W' < \partial_2 f(|x|, W) W'.$$

  If $\tilde{\varphi} \not\equiv 0$, it follows using $W' < 0$

  $$\|\tilde{\varphi}\|_{\dot{H}^s}^2 \geq \int_{\mathbb{R}^n} \tilde{\varphi}^2 \frac{((-\Delta)^s W)'}{W'} > \int_{\mathbb{R}^n} \tilde{\varphi}^2 \partial_2 f(|x|, W) = \|\tilde{\varphi}\|_{\dot{H}^s}^2, \quad \text{contradiction.}$$

  Thus $\tilde{\varphi} \equiv 0$.

Warm-up: Proving $\|\varphi\|_{\dot H^1}^2 \geq \int_{\mathbb{R}^n} \varphi^2 \frac{(-\Delta U)'}{U'}$ (if $\int_{B_R} \varphi = 0$ for all R)

Step 1: Integration by parts
If $\varphi = \eta V$, then

$$\int_{\mathbb{R}^n} |\nabla \varphi|^2 = \int_{\mathbb{R}^n} |\nabla(\eta V)|^2 = \int_{\mathbb{R}^n} \eta^2 V(-\Delta) V + \int_{\mathbb{R}^n} |\nabla \eta|^2 V^2.$$

Step 2: Using orthogonality
If additionally $\int_{B_R} \varphi = 0$ for all R, then

$$\int_{\mathbb{S}^{n-1}} |\nabla_\theta \eta(R\theta)|^2 \, \mathrm{d}\theta \geq (n-1) \int_{\mathbb{S}^{n-1}} \eta(R\theta) \, \mathrm{d}\theta \qquad \text{for all } R > 0.$$

If V is radial, it follows that $\int_{\mathbb{R}^n} |\nabla \eta|^2 V^2 \geq (n-1) \int_{\mathbb{R}^n} \frac{\eta^2}{r^2} V^2$.

Step 3: Conclusion
Choose $V = U'$. Since $((-\Delta) U)' = (-\Delta)(U') + \frac{n-1}{r^2} U'$, Steps 1 and 2 give

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla \varphi|^2 &\geq \int_{\mathbb{R}^n} \eta^2 U'(-\Delta) U' + (n-1) \int_{\mathbb{R}^n} \frac{\eta^2}{r^2} U'^2 \\ &= \int_{\mathbb{R}^n} \eta^2 U'(-\Delta U)' = \int_{\mathbb{R}^n} \varphi^2 \frac{(-\Delta U)'}{U'}. \quad \square \end{aligned}$$

Proving $\|\varphi\|^2_{\dot H^s} \geq \int_{\mathbb{R}^n} \varphi^2 \frac{((-\Delta)^s U)'}{U'}$ (if $\int_{B_R} \varphi = 0$ for all R)

Step 1: Integration by parts

If $\psi = \eta V$, then

$$\|\psi\|^2_{\dot H^s} = \|\eta V\|^2_{\dot H^s} = \int_{\mathbb{R}^n} \eta^2 V(-\Delta)^s V + \underbrace{\frac{C_{n,s}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{V(x)V(y)|\eta(x)-\eta(y)|^2}{|x-y|^{n+2s}}}_{=:I_V(\eta)}.$$

Step 2: Using orthogonality

Let $(A_k)_{k \in \mathbb{N}_0}$ be a spherical harmonics basis of $L^2(\mathbb{S}^{n-1})$ with $⨍_{\mathbb{S}^{n-1}} A_k = 1$. If $\int_{B_R} \varphi = 0$ for all R, then

$$\varphi = \sum_{k \geq 1} \varphi_k A_k \quad \text{for certain radial } \varphi_k, \text{ and} \quad \|\varphi\|^2_{\dot H^s} = \sum_{k \geq 1} \|\varphi_k A_k\|^2_{\dot H^s} \geq \|\varphi_k A_1\|^2_{\dot H^s}.$$

Step 3: Conclusion

Say $A_1 = \sqrt{n} \frac{x_1}{|x|}$. For any k, write

$$\varphi_k A_1 = \sqrt{n} \varphi_k \frac{x_1}{|x|} = \sqrt{n} \frac{\varphi_k}{U'} U' \frac{x_1}{|x|} := \sqrt{n} \eta_k \partial_1 U.$$

Applying Step 1 with $\eta = \eta_k$ and $V = \partial_1 U$, we get

$$\|\eta_k \partial_1 U\|^2_{\dot H^s} = \int_{\mathbb{R}^n} \eta_k^2 \partial_1 U (-\Delta)^s \partial_1 U + I_{\partial_1 U}(\eta_k).$$

Proving $\|\varphi\|_{\dot H^s}^2 \geq \int_{\mathbb{R}^n} \varphi^2 \frac{((-\Delta)^s U)'}{U'}$ (if $\int_{B_R} \varphi = 0$ for all R)

Step 3 (cont'd):

$$\|\eta_k \partial_1 U\|_{\dot H^s}^2 = \int_{\mathbb{R}^n} \eta_k^2 \partial_1 U (-\Delta)^s \partial_1 U + I_{\partial_1 U}(\eta_k).$$

Since U is radial decreasing, $\partial_1 U$ is antisymmetric wrt $\{x_1 = 0\}$ and $\partial_1 U(x) \leq 0$ if $x_1 > 0$.
Changing variables $x \mapsto \bar{x} = (-x_1, x_2, ..., x_n)$ and $y \mapsto \bar{y}$ gives

$$\frac{2}{C_{n,s}} I_{\partial_1 U}(\eta_k) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\partial_1 U(x) \partial_1 U(y) |\eta_k(x) - \eta_k(y)|^2}{|x-y|^{n+2s}}$$
$$= 2 \iint_{\{x_1 > 0, y_1 > 0\}} \partial_1 U(x) \partial_1 U(y) |\eta_k(x) - \eta_k(y)|^2 \left(\frac{1}{|x-y|^{n+2s}} - \frac{1}{|\bar{x} - y|^{n+2s}} \right) > 0.$$

Recalling $\eta_k = \frac{\varphi_k}{U'}$ and $\partial_1 U = U' \frac{x_1}{|x|}$,

$$\|\varphi_k A_1\|_{\dot H^s}^2 \geq n \int_{\mathbb{R}^n} \eta_k^2 \partial_1 U (-\Delta)^s \partial_1 U = n \int_{\mathbb{R}^n} \eta_k^2 \partial_1 U \partial_1 (-\Delta)^s U$$
$$= n \int_{\mathbb{R}^n} \varphi_k^2 \frac{((-\Delta)^s U)'}{U'} \frac{x_1^2}{|x|^2} = \int_{\mathbb{R}^n} \varphi_k^2 \frac{((-\Delta)^s U)'}{U'}.$$

By Step 2, summing over k gives the claim. □

Many thanks for your attention...

...und herzlichen Glückwunsch zur Habilitation, Konstantin!!!