Ground States of Spin Boson Models and Long Range Order in 1D Ising Models

Traces of the Ising Phase Transition in the Spin Boson Model

Benjamin Hinrichs

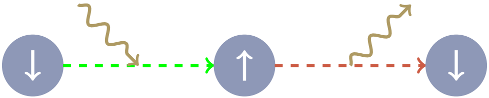
Paderborn University
Institute of Mathematics

Gauss Mini Workshop on Analysis, PDE, and Mathematical Physics
TU Braunschweig
January 23, 2025

joint work with Volker Betz, Mino Nicola Kraft and Steffen Polzer

Spin Boson Model

– describes a two-state system (spin) interacting with a quantum field



$$\boxed{H_{\mathsf{SB}}(\lambda) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z} \otimes \mathbb{1} + \mathbb{1} \otimes \mathsf{d}\Gamma(\omega) + \lambda \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x} \otimes (a(v) + a^\dagger(v))}$$

on $\mathbb{C}^2 \otimes \mathcal{F}$, where

– Fock space $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}$ direct sum over n-particle spaces $\mathcal{F}^{(n)}$,

with $\mathcal{F}^{(n)} = L^2_{\mathsf{sym}}(\mathbb{R}^{d\cdot n}) = \{f | f(k_1, \dots, k_n) = f(k_{\pi(1)}, \dots, k_{\pi(n)})\}$

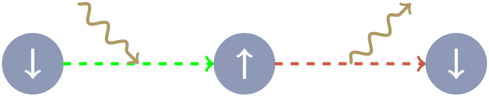
– free energy: $\mathsf{d}\Gamma(\omega) \restriction \mathcal{F}^{(n)} = \sum_{i=1}^n \omega(k_i)$

– annihilation: $a(v) : \mathcal{F}^{(n+1)} \to \mathcal{F}^{(n)}$,

$$a(v)f(k_1, \dots, k_n) = \sqrt{n+1} \int \overline{v(k)} f(k, k_1, \dots, k_n)$$

– creation: $a^\dagger(v) = a(v)^*$, CCR: $[a(f), a^\dagger(g)] = \langle f, g \rangle$.

Ground States in the Spin Boson Model

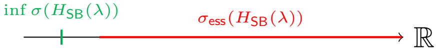


$$\boxed{H_{\mathsf{SB}}(\lambda) = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \lambda \sigma_x \otimes (a(v) + a^\dagger(v))}$$

– $\omega : \mathbb{R}^d \to [0, \infty)$ measurable, positive a.e.
– selfadjoint, lower-bounded operator if $v \in L^2(\mathbb{R}^d)$ and $\omega^{-1/2} v \in L^2(\mathbb{R}^d)$

Physical Example: $d = 3$, $\omega(k) = \sqrt{m^2 + |k|^2}$, $v(k) = \omega(k)^{-1/2} \mathbf{1}_{|k|<\Lambda}$ with UV cutoff $\Lambda \in (0, \infty)$

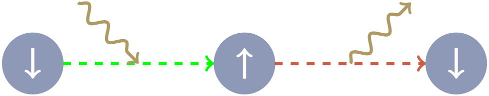
Arai, Hirokawa 1995: $\sigma_{\mathsf{ess}}(H_{\mathsf{SB}}(\lambda)) \subset [\inf \sigma(H_{\mathsf{SB}}(\lambda) + \underbrace{\operatorname{ess\,inf} \omega}_{=m}, \infty)]$



$\sigma(H_{\mathsf{SB}}(\lambda))$ in the case that ω is continuous and $\sup \omega = \infty$.

Is $\inf \sigma(H_{\mathsf{SB}}(\lambda))$ still an eigenvalue if $(m =) \operatorname{ess\,inf} \omega = 0$, i.e., does the spin boson model $H_{\mathsf{SB}}(\lambda)$ have a ground state?

Some History on the Ground State Problem



$$\boxed{H_{\mathsf{SB}}(\lambda) = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \lambda\sigma_x \otimes (a(v) + a^\dagger(v))}$$

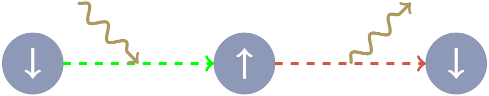
Massive Case. Arai, Hirokawa 1995

If $\operatorname{essinf} \omega > 0$, then $H_{\mathsf{SB}}(\lambda)$ has a ground state for all $\lambda \in \mathbb{R}$.

Infrared Regular Case. Arai, Hirokawa 1995, Bach, Fröhlich, Sigal 1998, Gérard 2000, Hirokawa, Hiroshima, Lőrinczi 2014

If $v/\omega \in L^2(\mathbb{R}^d)$, then $H_{\mathsf{SB}}(\lambda)$ has a ground state for all $\lambda \in \mathbb{R}$.

Some History on the Ground State Problem



$$\boxed{H_{\mathsf{SB}}(\lambda) = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \lambda\sigma_x \otimes (a(v) + a^\dagger(v))}$$

Massive Case. Arai, Hirokawa 1995

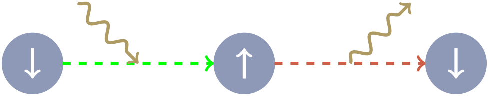
If $\operatorname{essinf} \omega > 0$, then $H_{\mathsf{SB}}(\lambda)$ has a ground state for all $\lambda \in \mathbb{R}$.

Infrared Regular Case. Arai, Hirokawa 1995, Bach, Fröhlich, Sigal 1998, Gérard 2000, Hirokawa, Hiroshima, Lőrinczi 2014

If $v/\omega \in L^2(\mathbb{R}^d)$, then $H_{\mathsf{SB}}(\lambda)$ has a ground state for all $\lambda \in \mathbb{R}$.

Infared Singular (and physical) Case.

Some History on the Ground State Problem



$$\boxed{H_{\mathsf{SB}}(\lambda) = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \lambda\sigma_x \otimes (a(v) + a^\dagger(v))}$$

Massive Case. Arai, Hirokawa 1995

If $\operatorname{essinf}\omega > 0$, then $H_{\mathsf{SB}}(\lambda)$ has a ground state for all $\lambda \in \mathbb{R}$.

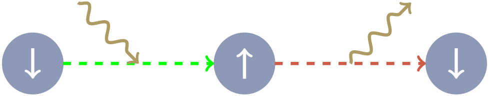
Infrared Regular Case. Arai, Hirokawa 1995, Bach, Fröhlich, Sigal 1998, Gérard 2000, Hirokawa, Hiroshima, Lőrinczi 2014

If $v/\omega \in L^2(\mathbb{R}^d)$, then $H_{\mathsf{SB}}(\lambda)$ has a ground state for all $\lambda \in \mathbb{R}$.

Infared Singular (and physical) Case. Hasler, Herbst 2011, Bach, Ballesteros, Könenberg, Menrath 2017

If $v/\omega \notin L^2(\mathbb{R}^d)$, then $H_{\mathsf{SB}}(\lambda)$ has a ground state for small $\lambda \in \mathbb{R}$.

Some History on the Ground State Problem



$$\boxed{H_{\mathsf{SB}}(\lambda) = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \lambda \sigma_x \otimes (a(v) + a^\dagger(v))}$$

Massive Case. Arai, Hirokawa 1995

If $\operatorname{essinf} \omega > 0$, then $H_{\mathsf{SB}}(\lambda)$ has a ground state for all $\lambda \in \mathbb{R}$.

Infrared Regular Case. Arai, Hirokawa 1995, Bach, Fröhlich, Sigal 1998, Gérard 2000, Hirokawa, Hiroshima, Lőrinczi 2014

If $v/\omega \in L^2(\mathbb{R}^d)$, then $H_{\mathsf{SB}}(\lambda)$ has a ground state for all $\lambda \in \mathbb{R}$.

Infared Singular (and physical) Case. Hasler, Herbst 2011, Bach, Ballesteros, Könenberg, Menrath 2017

If $v/\omega \notin L^2(\mathbb{R}^d)$, then $H_{\mathsf{SB}}(\lambda)$ has a ground state for small $\lambda \in \mathbb{R}$.

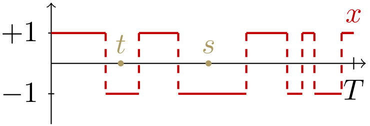
Hasler, H., Siebert 2021–22

$H_{\mathsf{SB}}(\lambda)$ has a ground state for $|\lambda| < \frac{1}{\sqrt{5}} \|\omega^{-1/2} v\|_2^{-1}$.

Continuous Ising Model

Jump Process X_t:

- Markov process
- state space $\{\pm 1\}$
- Exp(1)–distributed jump times
- uniformly distributed at $t = 0$



$W : \mathbb{R} \to [0, \infty)$ interaction function

Ising Energy: $\mathscr{E}_{W,T}(x) = -\int_0^T \int_0^T W(t-s) x_t x_s dt ds$

Partition Function: $\mathscr{Z}_{W,T} = \mathbb{E}\big[\exp\left(-\mathscr{E}_{W,T}(X)\right)\big]$

Expectation Values: $\langle\!\langle O \rangle\!\rangle_W = \lim_{T\to\infty} \frac{1}{\mathscr{Z}_{W,T}(\mu)} \mathbb{E}\big[O \exp\left(-\mathscr{E}_{W,T}(X)\right)\big]$

In the spirit of Ising 1954: $\langle\!\langle X_t X_s \rangle\!\rangle_0 = \mathrm{e}^{-2|t-s|}$

What is the asymptotic behavior of $\langle\!\langle X_t X_s \rangle\!\rangle_W$ as $|t-s| \to \infty$?

Short-Range: $W(t) \lesssim t^{-2-\varepsilon}$, Long-Range: $W(t) \geqslant t^{-2}$

Feynman–Kac Formula: Connecting the Models

Spin Boson Model.

$$\boxed{H_{\mathsf{SB}}(\lambda) = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \mathsf{d}\Gamma(\omega) + \lambda\sigma_x \otimes (a(v) + a^\dagger(v))}$$

Ising Model.

$$\boxed{\mathscr{Z}_{W,T} = \mathbb{E}\Big[\exp\Big(\int_0^T \int_0^T W(t-s) X_t X_s dt ds\Big)\Big]}$$

Feynman–Kac Formula.

If $W_\lambda(t) = \frac{\lambda^2}{8} \int |v(k)|^2 \mathsf{e}^{-|t|\omega(k)} \mathsf{d}k$, then

$$\langle \Omega_\downarrow, \mathsf{e}^{-tH_{\mathsf{SB}}(\lambda)} \Omega_\downarrow \rangle = \mathsf{e}^t \mathscr{Z}_{W_\lambda,t}.$$

Physical Example ($m = 0$): $W_\lambda(t) = \frac{1}{2}\pi\lambda^2\big(t^{-2} - \mathsf{e}^{-\Lambda t}(\Lambda t^{-1} + t^{-2})\big)$

A non-complete history.

Emery, Luther 1974, Spohn, Dümcke 1985, Fannes, Nachtergaele 1988, Spohn 1989, Abdessalam 2011, Hirokawa, Hiroshima, Lőrinczi 2014

Vaccum Overlap vs. Ising Correlations

Spin Boson Model.

$$\boxed{H_{\mathsf{SB}}(\lambda) = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \mathsf{d}\Gamma(\omega) + \lambda\sigma_x \otimes (a(v) + a^\dagger(v))}$$

Ising Model.

$$\boxed{\mathscr{Z}_{W,T} = \mathbb{E}\Big[\exp\Big(\int_0^T \int_0^T W(t-s)X_t X_s dtds\Big)\Big]}$$

Feynman–Kac Formula.

$$\langle \Omega_\downarrow, \mathsf{e}^{-tH_{\mathsf{SB}}(\lambda)}\Omega_\downarrow \rangle = \mathsf{e}^t \mathscr{Z}_{W_\lambda,t}, \; W_\lambda(t) = \frac{\lambda^2}{8}\int |v(k)|^2 \mathsf{e}^{-|t|\omega(k)} \mathsf{d}k$$

Strategy.

We will now rewrite $\rho(\lambda) := \langle \Omega_\downarrow, P_{H_{\mathsf{SB}}(\lambda)}(\{\inf \sigma(H_{\mathsf{SB}}(\lambda))\})\Omega_\downarrow \rangle$ in terms of Ising correlation functions, where $\Omega_\downarrow = \binom{0}{1} \otimes \Omega$ (ground state of $H_{\mathsf{SB}}(0)$).

Remark. $H_{\mathsf{SB}}(\lambda)$ has a ground state $\iff \rho(\lambda) > 0$. (Perron–Frobenius)

Vacuum Overlap vs. Ising Correlations

Theorem (Main Result). Betz, H., Kraft, Polzer 2025+

$$\rho(\lambda)^{-1} = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{4^n n!} \int_{[0,\infty)^n} \int_{[0,t]} \langle\!\langle X_0 X_{s_1} \cdots X_{s_n} \rangle\!\rangle_{W_\lambda} \langle\!\langle X_0 X_{t_1 - s_1} \cdots X_{t_n - s_n} \rangle\!\rangle_{W_\lambda} \mathrm{d}s W_\lambda(t_1) \cdots W_\lambda(t_n) \mathrm{d}t$$

Proof. $\rho(\lambda) = \lim_{T\to\infty} \mathscr{Z}^2_{W_\lambda,T} / \mathscr{Z}_{W_\lambda,2T}$ and a lengthy calculation. ∎

Corollary 1 (Monotonicity).

$\lambda \mapsto \rho(\lambda)$ is even and decreasing in $|\lambda|$.
Especially, there exists $\lambda_0 \in [0,\infty]$ such that $H_{\mathsf{SB}}(\lambda)$ has a ground state if $|\lambda| < \lambda_0$ and none if $|\lambda| > \lambda_0$.

Proof. Ising correlation inequalities (Griffiths–Kelly–Sherman) imply monotonicity in the interaction strength. ∎

$\rightarrow$ In the case $0 < \lambda_0 < \infty$, the spin boson model undergoes a phase transition (w.r.t. the coupling strength λ).

Existence of Ground States – Infrared Regular Case

Theorem (Main Result). Betz, H., Kraft, Polzer 2025+

$$\rho(\lambda)^{-1} = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{4^n n!} \int_{[0,\infty)^n} \int_{[0,t]} \langle\!\langle X_0 X_{s_1} \cdots X_{s_n} \rangle\!\rangle_{W_\lambda} \langle\!\langle X_0 X_{t_1 - s_1} \cdots X_{t_n - s_n} \rangle\!\rangle_{W_\lambda} \mathsf{d}s W_\lambda(t_1) \cdots W_\lambda(t_n) \mathsf{d}t$$

Corollary 2 (Infrared Regular Case).

If $\frac{v}{\omega} \in L^2(\mathbb{R}^d)$, then $H_{\mathsf{SB}}(\lambda)$ has a ground state for all $\lambda \in \mathbb{R}$, i.e., $\lambda_0 = \infty$.

Proof. $\int_{[0,t]} \langle\!\langle X_0 X_{s_1} \cdots X_{s_n} \rangle\!\rangle_{W_\lambda} \langle\!\langle X_0 X_{t_1 - s_1} \cdots X_{t_n - s_n} \rangle\!\rangle_{W_\lambda} \mathsf{d}s \leq t_1 \cdots t_n$ combined with $\frac{v}{\omega} \in L^2(\mathbb{R}^d) \iff \int_0^\infty t W_\lambda(t) < \infty$. ∎

This result was (as discussed before) already previously well-established.

Bach, Fröhlich, Sigal 1998
Gérard 2000, Dam, Møller 2020
Hirokawa, Hiroshima, Lőrinczi 2014

But: This does not cover the physical model!

Existence of Ground States – Infrared Singular Case

Theorem (Main Result). Betz, H., Kraft, Polzer 2025+

$$\rho(\lambda)^{-1} = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{4^n n!} \int_{[0,\infty)^n} \int_{[0,t]} \langle\!\langle X_0 X_{s_1} \cdots X_{s_n} \rangle\!\rangle_{W_\lambda} \langle\!\langle X_0 X_{t_1 - s_1} \cdots X_{t_n - s_n} \rangle\!\rangle_{W_\lambda} \mathrm{d}s W_\lambda(t_1) \cdots W_\lambda(t_n) \mathrm{d}t$$

Corollary 3 (Infrared Critical Case: Long Range Order $\implies$ Absence).

If $\frac{v}{\omega} \notin L^2(\mathbb{R}^d)$, i.e., $\int_0^\infty t W_\lambda(t) \mathrm{d}t = \infty$, then $\inf_{t \geq 0} \langle\!\langle X_0 X_t \rangle\!\rangle_{W_\lambda} > 0$ implies that $H_{\mathsf{SB}}(\lambda)$ has no ground state.

Proof. $\int_0^t \langle\!\langle X_0 X_s \rangle\!\rangle_{W_\lambda} \langle\!\langle X_0 X_{t-s} \rangle\!\rangle_{W_\lambda} \mathrm{d}s \geq t(\inf_{t\geq 0} \langle\!\langle X_0 X_t \rangle\!\rangle_{W_\lambda})^2$, so already the $n = 1$ summand is infinite. ■

Theorem (Ising Phase Transition). Betz, H., Kraft, Polzer 2025+

If $W(t) \geq c(1+t)^{-2}$ as $t \to \infty$ and c is sufficiently large, then $\inf_{t\geq 0} \langle\!\langle X_0 X_t \rangle\!\rangle_W > 0$. Especially, in the physical model, $\lambda_0 < \infty$.

Proof. Follows from results for discrete long range percolation models and a discretization procedure. Newman, Schulman 1986, Spohn 1989 ■

Summary on the Large Coupling Infrared Singular Case

Corollary 3.
If $\frac{v}{\omega} \notin L^2(\mathbb{R}^d) \iff \int_0^\infty t W_\lambda(t) \mathrm{d}t = \infty$, then $\inf_{t \geq 0} \langle\!\langle X_0 X_t \rangle\!\rangle_{W_\lambda} > 0$ implies that $H_{\mathsf{SB}}(\lambda)$ has no ground state.

Theorem (Ising Phase Transition). This is the case for λ sufficiently large if $W_1(t) \geqslant t^{-2}$ as $t \to \infty$ (recall $W_\lambda = \lambda^2 W_1$).

That is $\lambda_0 < \infty$ in the physical / infrared-critical case.

Betz, H., Kraft, Polzer 25[+]

An Existence Criterion

Corollary 3.
If $\frac{v}{\omega} \notin L^2(\mathbb{R}^d) \iff \int_0^\infty tW_\lambda(t)\mathrm{d}t = \infty$, then $\inf_{t\geq 0}\langle\!\langle X_0 X_t \rangle\!\rangle_{W_\lambda} > 0$ implies that $H_{\mathsf{SB}}(\lambda)$ has no ground state.

Theorem (Ising Phase Transition). This is the case for λ sufficiently large if $W_1(t) \geqslant t^{-2}$ as $t \to \infty$ (recall $W_\lambda = \lambda^2 W_1$).

That is $\lambda_0 < \infty$ in the physical / infrared-critical case.
Betz, H., Kraft, Polzer 25$^+$

Theorem (Lower Bound GS Vacuum Overlap). H., Polzer 25$^+$

$$\rho(\lambda) \geq \limsup_{T\to\infty} \exp\left(-\frac{1}{T}\int_{[0,T)^2} |t-s| W_\lambda(t-s) \langle\!\langle X_s X_t \rangle\!\rangle_{W_\lambda, T} \mathrm{d}s\mathrm{d}t\right)$$

Proof. $\rho(\lambda) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\mathscr{Z}_{W_\lambda,s}\mathscr{Z}_{W_\lambda,t-s}}{\mathscr{Z}_{W_\lambda,t}}\mathrm{d}s$ and Jensen's inequality ∎

All (up to Date) on the Infrared Singular Model

Existence of GS at Small Coupling. H., Polzer 2025+

$$\limsup_{T\to\infty} \int_{[0,T)^2} |t-s| W_\lambda(t-s) \langle\!\langle X_s X_t \rangle\!\rangle_{W_\lambda,T} \mathsf{d}s\mathsf{d}t < \infty$$

$$\implies H_{\mathsf{SB}}(\lambda) \text{ has a ground state.}$$

Theorem. Hasler, H., Siebert 2022

$$\limsup_{T\to\infty} \int_{[0,T)^2} \langle\!\langle X_s X_t \rangle\!\rangle_{W_\lambda,T} \mathsf{d}s\mathsf{d}t < \infty \text{ if } |\lambda| < \|W_\lambda\|_1 = 5^{-1/2} \|\omega^{-1/2} v\|_2^{-1}$$

Absence of GS at Large Coupling. Betz, H., Kraft, Polzer 2025+

$\int_0^\infty t W_\lambda(t) \mathsf{d}t = \infty,\ \inf_{t\geq 0} \langle\!\langle X_0 X_t \rangle\!\rangle_{W_\lambda} > 0$

$$\implies H_{\mathsf{SB}}(\lambda) \text{ has no ground state.}$$

These assumptions hold for large λ if $W_1 \geq t^{-2}$ as $t \to \infty$.

All (up to Date) on the Infrared Singular Model

Existence of GS at Small Coupling. H., Polzer 2025$^+$

$$\limsup_{T\to\infty} \int_{[0,T)^2} |t-s| W_\lambda(t-s) \langle\!\langle X_s X_t \rangle\!\rangle_{W_\lambda,T} \mathrm{d}s\mathrm{d}t < \infty$$

$$\implies H_{\mathsf{SB}}(\lambda) \text{ has a ground state.}$$

Theorem. Hasler, H., Siebert 2022

$$\limsup_{T\to\infty} \int_{[0,T)^2} \langle\!\langle X_s X_t \rangle\!\rangle_{W_\lambda,T} \mathrm{d}s\mathrm{d}t < \infty \text{ if } |\lambda| < \|W_\lambda\|_1 = 5^{-1/2} \|\omega^{-1/2} v\|_2^{-1}$$

Absence of GS at Large Coupling. Betz, H., Kraft, Polzer 2025$^+$

$\int_0^\infty t W_\lambda(t) \mathrm{d}t = \infty,\ \inf_{t\geq 0} \langle\!\langle X_0 X_t \rangle\!\rangle_{W_\lambda} > 0$

$$\implies H_{\mathsf{SB}}(\lambda) \text{ has no ground state.}$$

These assumptions hold for large λ if $W_1 \geq t^{-2}$ as $t \to \infty$.

Are these the only two cases? Aizenman, Barsky, Fernández 1987
Are the critical coupling in Ising and spin boson model identical?

All (up to Date) on the Infrared Singular Model

Existence of GS at Small Coupling. H., Polzer 2025$^+$

$$\limsup_{T\to\infty} \int_{[0,T)^2} |t-s| W_\lambda(t-s) \langle\!\langle X_s X_t \rangle\!\rangle_{W_\lambda,T} \mathrm{d}s\mathrm{d}t < \infty$$

$\implies H_{\mathsf{SB}}(\lambda)$ has a ground state.

Theorem. Hasler, H., Siebert 2022

$$\limsup_{T\to\infty} \int_{[0,T)^2} \langle\!\langle X_s X_t \rangle\!\rangle_{W_\lambda,T} \mathrm{d}s\mathrm{d}t < \infty \text{ if } |\lambda| < \|W_\lambda\|_1 = 5^{-1/2} \|\omega^{-1/2} v\|_2^{-1}$$

Absence of GS at Large Coupling. Betz, H., Kraft, Polzer 2025$^+$

$\int_0^\infty t W_\lambda(t)\mathrm{d}t = \infty,\ \inf_{t\geq 0} \langle\!\langle X_0 X_t \rangle\!\rangle_{W_\lambda} > 0$

$\implies H_{\mathsf{SB}}(\lambda)$ has no ground state.

These assumptions hold for large λ if $W_1 \geq t^{-2}$ as $t \to \infty$.

Are these the only two cases? Aizenman, Barsky, Fernández 1987

Are the critical coupling in Ising and spin boson model identical?

CONGRATULATIONS PD DR. KONSTANTIN MERZ!

THANK YOU FOR THE ATTENTION!