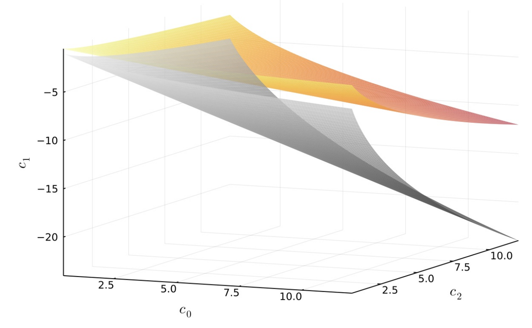
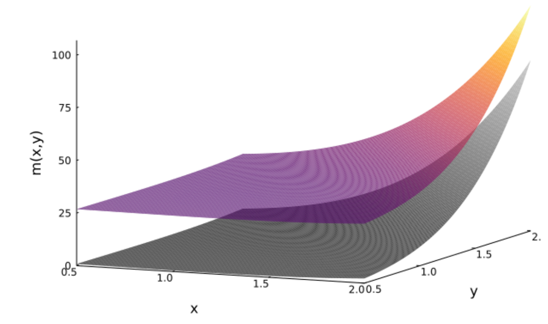


The Duality of SONC

Janin Heuer and Timo de Wolff



Problem Setting

Exponential sums on A: $\mathbb{R}^A = \mathrm{span}_{\mathbb{R}}\left(\left\{e^{\langle x,\alpha\rangle} : \alpha \in A\right\}\right)$

Support decomposition: $A = A^+ \cup A^-$, where $A^\pm = \{\alpha \in A : \pm c_\alpha > 0\}$

Goal: Find $\inf_{x\in\mathbb{R}^n} f(x)$, where $f(x) = \sum_{\alpha\in A} c_\alpha e^{\langle x,\alpha\rangle} \in \mathbb{R}^A$ by investigating

$$\sup\{\gamma \in \mathbb{R} \;:\; f(x) - \gamma \geq 0 \text{ for all } x \in \mathbb{R}^n\}.$$

☹ NP-hard problem in general

💡 Use *Certificates of Nonnegativity*!

Basics

Circuit functions:
- supported on minimally affine dependent set A
- satisfy $\alpha \in \mathrm{Vert}(\mathrm{conv}(A)) \Rightarrow c_\alpha > 0$.

Theorem [Iliman, de Wolff '16]
A circuit function $f = \sum_{\alpha\in A^+} c_\alpha e^{\langle x,\alpha\rangle} + c_\beta e^{\langle x,\beta\rangle}$ is nonnegative iff

$$|c_\beta| \leq \prod_{\alpha\in A^+}\left(\frac{c_\alpha}{\lambda_\alpha}\right)^{\lambda_\alpha} =: \Theta_f,$$

where the $\lambda_\alpha \in (0,1)$ are barycentric coordinates of β w.r.t. A^+ and Θ_f is the *circuit number* of f.

The SONC Cone

$$\mathcal{S}_A = \mathrm{cone}\left\{f \in \mathbb{R}^A : f \text{ nonnegative circuit function}\right\}$$

$$\mathcal{S}_{A^+,A^-} = \sum_{\beta\in A^-} \mathcal{S}_{A^+,\beta}$$

The Dual SONC Cone

$$\check{\mathcal{S}}_{A^+,A^-} \overset{\text{Thm.}}{=} \left\{ v \in \check{\mathbb{R}}^A : \begin{array}{c} v_\alpha \geq 0 \;\forall\, \alpha \in A^+, \text{ and} \\ \forall\, \beta \in A^- \exists \tau \in \mathbb{R}^n \text{ such that} \\ \ln\left(\frac{|v_\beta|}{v_\alpha}\right) \leq (\alpha-\beta)^\top \tau \;\forall\, \alpha \in A^+ \end{array} \right\}$$

$$= \bigcap_{\beta\in A^-} \check{\mathcal{S}}_{A^+,\beta}$$

New: The DSONC Cone

Let $f(x) = \sum_{\alpha\in A^+} c_\alpha e^{\langle x,\alpha\rangle} + c_\beta e^{\langle x,\beta\rangle}$. If f is a circuit function then we define the

Dual Circuit Number: $\prod_{\alpha\in A^+} c_\alpha^{\lambda_\alpha} =: \check{\Theta}_f$.

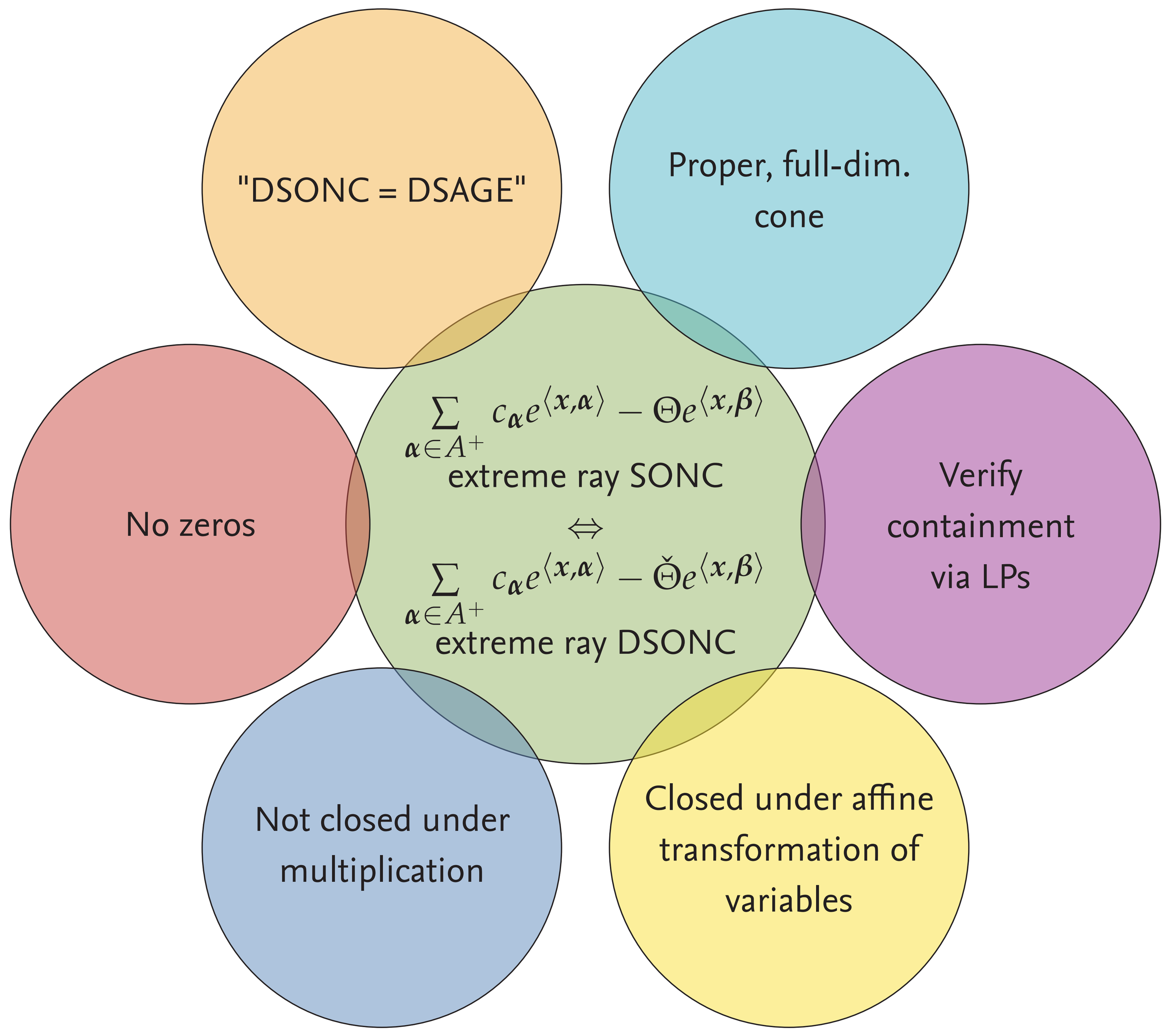
$$\mathcal{D}_{A^+,\beta} = \left\{f \in \mathbb{R}^A : \boldsymbol{c} \in \check{\mathcal{S}}_{A^+,\beta}\right\}$$

$$\overset{\text{Thm.}}{=} \left\{f \in \mathbb{R}^A : f \text{ circuit function with } |c_\beta| \leq \check{\Theta}_f\right\}$$

$$\mathcal{D}_{A^+,A^-} = \sum_{\beta\in A^-} \mathcal{D}_{A^+,\beta}$$

If $\#A^- > 1$: $\check{\mathcal{S}}_{A^+,A^-} \subset \check{\mathcal{S}}_{A^+,\bar{\beta}} \cong \mathcal{D}_{A^+,\bar{\beta}} \subset \mathcal{D}_{A^+,A^-} \subset \mathcal{S}_{A^+,A^-}$ holds for all $\bar{\beta} \in A^-$.

DSONC: Results (Informal Collection)



DSONC: Equilibria & Algebraic Viewpoint

Consider $f = \sum_{i=1}^n c_i e^{\langle x,\alpha_i\rangle} + c_\beta e^{\langle x,\beta\rangle}$ circuit function, and

Equilibrium $\mathrm{eq}(f)$: unique point satisfying $c_1 e^{\langle \mathrm{eq}(f),\alpha_1-\beta\rangle} = \cdots = c_n e^{\langle \mathrm{eq}(f),\alpha_n-\beta\rangle}$

Exponential toric morphism: $\varphi_A : \mathbb{R}^n \to \mathbb{R}^{d+1}$, $\varphi_A(x) = \left(e^{\langle x,\alpha_1\rangle}, \ldots, e^{\langle x,\alpha_d\rangle}\right)$

Group action: $G : \mathbb{R}^n \times \mathbb{R}^A \to \mathbb{R}^A$, $(w, f) \mapsto f(x - w)$

Thm.: f in boundary of DSONC cone iff $|c_\beta| = c_i e^{\langle \mathrm{eq}(f),\alpha_i-\beta\rangle}$ holds for all $i \in [n]$.

Thm.: $\mathcal{D}_{A^+,\beta} = \{G(w, \langle\varphi_A(x), t\cdot(\mathbf{1},-1)\rangle) \in \mathbb{R}^A : w \in \mathbb{R}^n, t \in \mathbb{R}_{>0}\}$

Interpretation: **DSONCs are "circuit functions with equilibrium $(\mathbf{0},\mathbf{0},\ldots,\mathbf{0})$ plus group action".**

Compare: *SONCs are "circuit functions with singular point $(0,0,\ldots,0)$ plus group action".*

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arXiv:2204.03918