Global Optimisation via the dual SONC cone & linear Programming

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Goal

Minimise exponential sums with finite support $A \subset \mathbb{R}^n$, coefficient vector $c \in \mathbb{R}^A$ and $x \in \mathbb{R}^n$.

$$f(x) = \sum_{\alpha \in A} c_\alpha e^{\langle x, \alpha \rangle}$$

Special case $A \subset \mathbb{N}^n$: Polynomials on $\mathbb{R}^n_{>0}$.

Problems

- $\min_{x \in \mathbb{R}^n} f(x)$ is not convex.
- Solving $\min_{x \in \mathbb{R}^n} f(x)$ is NP-hard in general.

Solution

Solving $\min_{x \in \mathbb{R}^n} f(x)$ is equivalent to solving

$$\max \{\gamma \in \mathbb{R} : f(x) - \gamma \geq 0 \text{ for all } x \in \mathbb{R}^n\}.$$

Tool for finding lower bounds:

Certificates of Nonnegativity

- imply nonnegativity,
- hold for a large class of exponential sums/polynomials,
- are easier to test.

Certificate Examples

Sum of Squares (SOS)

If f can be written as SOS, f is nonnegative.

Requires semidefinite programming.
Hilbert (1888): Not all nonnegative polynomials are SOS.

Sum of nonnegative Circuits (SONC)

Reznick (1989), Iliman & de Wolff (2016): Use AM-GM inequality to certify nonnegativity.

Requires geometric/relative entropy programming.
Chandrasekaran & Shah (2016): similar approach, developed independently

Nonnegative Circuits

Definition

A function $f(x) = \sum_{\alpha \in A} c_\alpha e^{\langle x, \alpha \rangle} \in \mathbb{R}^A$ is called a circuit if

(1) $\text{supp}(f)$ is minimally affine dependent,

(2) $\text{conv}(\text{supp}(f))$ is a simplex, and

(3) $\alpha \in \text{Vert}(\text{conv}(\text{supp}(f)) \Longrightarrow c_\alpha > 0$.

Note

Nonnegativity of circuit functions can be decided by an invariant called circuit number [Iliman, de Wolff (2016)]

SONC cone

for fixed support

$\mathcal{S}_A = \{f \in \mathbb{R}^A : f \text{ is SONC}\}$

Part II: Outline

Consider SONC cone for fixed support set and **fixed sign distribution** $A = A^+ \cup A^- \subset \mathbb{R}^n$, where $A^\pm = \{\alpha \in A : \pm c_\alpha > 0\}$

$$\mathcal{S}_{A^\pm} = \left\{ f \in \mathbb{R}^{A^+ \cup A^-} : f \text{ is SONC} \right\}.$$

Fact	$\mathcal{S}_{A^\pm} \subset \left\{ f \in \mathbb{R}^{A^+ \cup A^-} : f \geq 0 \right\}$
Our Contribution	Consider **dual** SONC cone $\check{\mathcal{S}}_{A^\pm}$ and show: • Containment in $\check{\mathcal{S}}_{A^\pm}$ is Certificate of Nonnegativity. • Optimising over $\check{\mathcal{S}}_{A^\pm}$ is **linear program.**

The Dual SONC Cone

For $f \in \mathbb{R}^A$ with coefficient vector $c \in \mathbb{R}^A$ consider the natural duality pairing

$$v(f) \;=\; \sum_{\alpha \in A^+} v_\alpha c_\alpha + \sum_{\beta \in A^-} v_\beta c_\beta$$

and for every $v \in \check{\mathbb{R}}^A$ associate

$$f \;=\; \sum_{\alpha \in A^+} v_\alpha e^{\langle x, \alpha \rangle} + \sum_{\beta \in A^-} v_\beta e^{\langle x, \beta \rangle}.$$

Dual (signed) SONC cone

$$\check{\mathcal{S}}_{A^\pm} \;=\; \left\{ v \in \check{\mathbb{R}}^A : v(f) \geq 0 \;\forall\, f \in \mathcal{S}_{A^\pm} \right\}$$

Dual Cone contained in Primal

Using a particular representation of $\check{\mathcal{S}}_{A^\pm}$, we can show:

Proposition

$$\check{\mathcal{S}}_{A^\pm} \subset \mathcal{S}_{A^\pm}$$

$\longrightarrow$ containment in $\check{\mathcal{S}}_{A^\pm}$ is certificate of nonnegativity.

Optimising over Dual Cone is LP

Using a different representation of $\check{\mathcal{S}}_{A^\pm}$ yields:

Proposition

The following linear feasibility program in $\#A^-$ variables $(\tau^{(\beta)})_{\beta \in A^-}$ verifies containment in $\check{\mathcal{S}}_{A^\pm}$:

$$\ln\left(\frac{|v_\beta|}{v_\alpha}\right) \leq (\alpha - \beta)^T \tau^{(\beta)} \;\forall\, \beta \in A^-,\; \alpha \in A^+$$

Recall:

Want to solve

SONC $\Rightarrow \geq \ 0$
Dual contained in primal

$$\text{min}\,\{\gamma \in \mathbb{R} \ : \ f(x) + \gamma \geq 0\}$$
$$\leq \text{min}\,\{\gamma \in \mathbb{R} \ : \ f(x) + \gamma \in \mathcal{S}_{A\pm}\}$$
$$\leq \text{min}\,\left\{\check{\gamma} \in \mathbb{R} \ : \ v + \check{\gamma} e_0 \in \check{\mathcal{S}}_{A\pm}\right\}$$

Optimising over Dual Cone is LP

$$\check{\gamma}^* \;=\; \min\left\{\check{\gamma} \in \mathbb{R} \;:\; v + \check{\gamma} e_0 \in \check{\mathcal{S}}_{A^\pm}\right\}$$

Theorem

The following LP in variables $(\tau^{(\beta)})_{\beta \in A^-}$ *and* $c = \ln(|v_0 + \check{\gamma}|)$ *returns the lower bound* $-\check{\gamma}^* = v_0 - e^c$ *for* $-\check{\gamma} \neq v_0$*.*

$$\begin{array}{lll} & \min \; c & \\ & (1) & \forall\, \beta \in A^-, \forall \alpha \in A^+ \setminus \{0\} : \\ \textit{s.t.} & & \ln\left(\frac{|v_\beta|}{v_\alpha}\right) \leq (\alpha - \beta)^T \tau^{(\beta)}, \\ & (2) & \ln\left(|v_\beta|\right) - c \leq (-\beta)^T \tau^{(\beta)} \;\; \forall\, \beta \in A^- \end{array}$$

Conclusion

- Effective algorithm for optimizing over the dual SONC cone.
- Computationally more stable than existing methods.
- Yields, in general, worse results than the SOS, SONC, and SAGE approach.
- Promising runtimes.
- Gives a result whenever a solution in the dual cone exists.

References

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Thank you for your Attention!