Global Optimization via the dual SONC cone & linear Programming

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Mareike Dressler Janin Heuer
Helen Naumann Timo de Wolff

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Technische Universität Braunschweig

Goal

Minimise exponential sums with finite support $A \subset \mathbb{R}^n$, coefficient vector $c \in \mathbb{R}^A$ and $x \in \mathbb{R}^n$.

$$f(x) = \sum_{\alpha \in A} c_\alpha e^{\langle x, \alpha \rangle}$$

Special case $A \subset \mathbb{N}^n$: Polynomials on $\mathbb{R}^n_{>0}$.

Problems

- $\min_{x \in \mathbb{R}^n} f(x)$ is not convex.
- Solving $\min_{x \in \mathbb{R}^n} f(x)$ is NP-hard in general.

Solution

Solving $\min_{x \in \mathbb{R}^n} \; f(x)$ is equivalent to solving

$$\max \{\gamma \in \mathbb{R} \,:\, f(x) - \gamma \geq 0 \text{ for all } x \in \mathbb{R}^n\}.$$

Tool for finding lower bounds:

Certificates of Nonnegativity

- imply nonnegativity,
- hold for a large class of exponential sums/polynomials,
- are easier to test than nonnegativity.

Certificate Examples

Sum of Squares (SOS)

If f can be written as SOS, f is nonnegative.

Requires semidefinite programming.

Sum of nonnegative Circuits (SONC)

Reznick (1989), Iliman & de Wolff (2016): If f can be written as SONC, f is nonnegative.

Requires geometric/relative entropy programming.
One interpretation: Use AM-GM inequality to certify nonnegativity.

$$\frac{1}{n}\sum_{i=1}^{n} t_i \geq \sqrt[n]{\prod_{i=1}^{n} t_i} \Longrightarrow \frac{x^4y^2 + x^2y^4 + 1}{3} \geq \sqrt[3]{x^6y^6}$$

$$\Longrightarrow x^4y^2 + x^2y^4 - 3x^2y^2 + 1 \geq 0$$

Outline

Remember | $f(x) \;=\; \sum_{\alpha \in A} c_\alpha e^{\langle x, \alpha \rangle}$

Consider SONC cone for fixed support set and **fixed sign distribution** $A = A^+ \cup A^- \subset \mathbb{R}^n$, where $A^\pm = \{\alpha \in A : \pm c_\alpha > 0\}$

$$\begin{aligned} \mathcal{S}_{A^\pm} \;&=\; \left\{ f \in \mathbb{R}^{A^+ \cup A^-} : f \text{ is SONC} \right\} \\ &\subset \left\{ f \in \mathbb{R}^{A^+ \cup A^-} : f \geq 0 \right\}. \end{aligned}$$

Our Contribution | Consider **dual** SONC cone $\check{\mathcal{S}}_{A^\pm}$ and show:

- Containment in $\check{\mathcal{S}}_{A^\pm}$ is certificate of nonnegativity.
- Optimizing over $\check{\mathcal{S}}_{A^\pm}$ is **linear program.**

The Dual SONC Cone

For $f \in \mathbb{R}^A$ with coefficient vector $c \in \mathbb{R}^A$ consider the natural duality pairing

$$v(f) = \sum_{\alpha \in A^+} v_\alpha c_\alpha + \sum_{\beta \in A^-} v_\beta c_\beta.$$

Dual (signed) SONC cone

$$\check{\mathcal{S}}_{A^\pm} = \left\{ v \in \check{\mathbb{R}}^A : v(f) \geq 0 \; \forall \, f \in \mathcal{S}_{A^\pm} \right\}$$

For every $v \in \check{\mathbb{R}}^A$ associate

$$f = \#A^- \cdot \sum_{\alpha \in A^+} v_\alpha e^{\langle x, \alpha \rangle} + \sum_{\beta \in A^-} v_\beta e^{\langle x, \beta \rangle}.$$

Dual Cone contained in Primal Cone

Using a particular representation of $\check{\mathcal{S}}_{A^\pm}$, we can show:

Proposition

$$\check{\mathcal{S}}_{A^\pm} \subset \mathcal{S}_{A^\pm}$$

$\longrightarrow$ containment in $\check{\mathcal{S}}_{A^\pm}$ is certificate of nonnegativity.

Optimizing over Dual Cone is LP

Using a different representation of $\check{\mathcal{S}}_{A^\pm}$ yields:

Theorem

Let $f = \#A^- \cdot \sum_{\alpha \in A^+} v_\alpha e^{\langle x, \alpha \rangle} + \sum_{\beta \in A^-} v_\beta e^{\langle x, \beta \rangle}$.
The following linear feasibility program in $\#A^-$ variables $(\tau^{(\beta)})_{\beta \in A^-}$ verifies containment in $\check{\mathcal{S}}_{A^\pm}$:

$$\ln\left(\frac{|v_\beta|}{v_\alpha}\right) \leq (\alpha - \beta)^T \tau^{(\beta)} \; \forall \, \beta \in A^-, \; \alpha \in A^+$$

Result

Finding max $\{\gamma \in \mathbb{R} \; : \; f(x) - \gamma \in \check{\mathcal{S}}_{A^\pm}\}$ can be done by solving an LP.
$\longrightarrow$ Lower bound for max $\{\gamma \in \mathbb{R} \; : \; f(x) - \gamma \geq 0\}$.

Conclusion

- Effective algorithm for optimizing over the dual SONC cone.
- Computationally more stable than existing methods as it only requires linear programming.
- Yields, in general, worse bounds than preexisting approaches, since we are only optimizing over a subcone of the primal SONC cone.
- Promising runtimes.

References

Thank you for your attention!