Nullspace Conditions for Block-Sparse Recovery of Semidefinite Systems

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Problem setting

Consider a linear transformation $\mathcal{A}\colon \mathcal{S}^n \to \mathbb{R}^m$ given by

$$\mathcal{A}(X) = (A_1 \bullet X, \dots, A_m \bullet X)^\top,$$

where $A_1, \dots, A_m \in \mathcal{S}^n$ are in *block-diagonal form* w.r.t. blocks $B_1, \dots, B_k$.

Goal: Finding solutions $X \succeq 0$ of $\mathcal{A}(X) = b$ with minimal block support $\mathrm{BS}(X) := \{i \in [k] \,:\, X_{B_i} \neq 0\}$.

Formulation as optimization problem:

$$\min\{\|X\|_{*,0} \,:\, \mathcal{A}(X) = b, X \succeq 0\}, \tag{1}$$

with

$$\|X\|_{*,0} = \|(\|X_{B_1}\|_*, \dots, \|X_{B_k}\|_*)^\top\|_0,$$

where $\|\cdot\|_*$ denotes the nuclear norm $\|A\|_* = \sum_{i=1}^n \sigma_i(A)$ with singular values $\sigma_i(A)$ of A and

$$\|x\|_0 := |\mathrm{supp}(x)| = |\{i \in [n] \,:\, x_i \neq 0\}|.$$

Alternatively, analogous problems without the restriction $X \succeq 0$ can and have been studied.

Idea: replace non-convex $\|\cdot\|_{*,0}$ and consider convex relaxation

$$\min\{\|X\|_{*,1} \,:\, \mathcal{A}(X) = b, X \succeq 0\} \tag{2}$$

Aim: Characterize, when solving (2) yields a solution of (1) using *null space properties (NSPs)*

Motivation: Signal recovery, identifying minimal irreducible subsystems of spectrahedra

Special Case: Linear Problems

Consider 1×1-blocks. (1) without additional constraint $X \succeq 0$ reduces to well-studied vector case

$$\min\{\|x\|_0 \,:\, Ax = b,\ x \in \mathbb{R}^n\}. \tag{3}$$

Standard result: LP relaxation

$$\min\{\|x\|_1 \,:\, Ax = b,\ x \in \mathbb{R}^n\} \tag{4}$$

satisfies the NSP: Any s-sparse vector x is the unique solution of (4) iff for all $S \subset [n]$ with $|S| \leq s$ we have

$$\|v_S\|_1 < \|v_{\overline{S}}\|_1 \text{ for all } v \in \ker(A)\setminus\{0\}.$$

Analogously, (1) and (2) specialize to well studied nonnegative versions of (3) and (4) with corresponding non- negative NSP, see e.g. *Donoho, Foucart/Rauhut.*

Semidefinite Block-Matrix NSP

Definition. A linear transformation $\mathcal{A}(X)$ in block-diagonal form satisfies the *semidefinite block-matrix null space property* of order s iff for all $S \subset [k]$, $|S| \leq s$ and all $V \in \ker(\mathcal{A})\setminus\{0\}$, $V \in \mathcal{S}^n$ with $\lambda_j(V_{B_i}) \leq 0$ for all $j \in [n_i]$ with $i \in \overline{S}$, where n_i denotes the size of the block B_i,

$$\sum_{i\in S}\sum_{j\in[n_i]} \lambda_j(V_{B_i}) < \sum_{i\in\overline{S}}\sum_{j\in[n_i]} |\lambda_j(V_{B_i})| \tag{NSP$^*_{*,1,\succeq 0}$}$$

holds.

Theorem. *Let $\mathcal{A}(X)$ be a linear transformation in block-diagonal form, $b \in \mathbb{R}^m$ and $s \geq 1$. The following statements are equivalent:*

(i) If $\mathcal{A}(X) = b$ has a solution $X^0 \in \mathcal{S}^n_+$ with $\|X^0\|_{*,0} \leq s$, X^0 is the unique solution of (2).

(ii) $\mathcal{A}(X)$ satisfies the semidefinite block-matrix null space property of order s.

Special Case: Linear Block-Problems

If $A_1, \dots, A_m$ are diagonal matrices, (1) also captures the as of yet unstudied block vector setting for the signed case.

Corollary. Choosing (X) in diagonal form, above Theorem also shows that block-sparse solutions of linear systems

$$\begin{aligned} b &= Ax = [A[1] \cdots A[k]]\, x \\ x &\geq 0, \end{aligned} \tag{5}$$

where $A \in \mathbb{R}^{d\times n}$ consists of k blocks $A[i] \in \mathbb{R}^{d\times n_i}$ and $b \in \mathbb{R}^d$, can successfully be recovered by solving a linear program. More specifically, the following statements for the block-linear system (5) are equivalent:

(i) If $Ax = b$ has a nonnegative solution $x^0 \in \mathbb{R}^n$ with $\|x^0\|_{1,0} \leq s$, then x^0 is the unique solution of

$$\min\{\|x\|_{1,1} \,:\, Ax = b,\ x \geq 0\}.$$

(ii) A satisfies the nonnegative block-linear null space property of order s, i.e.,

$$\sum_{i\in S}\sum_{j\in B_i} v_j < \sum_{i\in\overline{S}}\sum_{j\in B_i} |v_j| \tag{NSP$_{1,1,\geq 0}$}$$

holds for all $S \subset [k]$, $|S| \leq s$ and all $v \in \ker(A)\setminus\{0\}$, with $v_j \leq 0$ for all $j \in B_i$ with $i \in \overline{S}$.

Outlook

- It can be shown that using positive semidefiniteness/nonnegativity captures cases, that can not be recovered by corresponding unrestricted NSPs.
- It is possible to formulate a more general framework unifying all the above mentioned cases.

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