Classifying Cubic Surfaces over Finite Fields with Orbiter

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1 Introduction

Orbiter [2, 1] can classify cubic surfaces with 27 lines over finite fields $\mathbb{F}_q$ under the collineation group $\mathrm{P\Gamma L}(4, q)$. Several different algorithms to solve this problem are available in Orbiter. One approach is described in [4] and relies on Schlaefli's notion of a double six as a substructure [8]. Another is the lifting of six-arcs in a plane, following [3] and in [6]. Earlier work on the classification of cubic surfaces over finite fields is Sadeh [7] and Hirschfeld [5]. Orbiter is a C++ class library that comes with a suite of command line applications. It is open source and available on GitHub[2]. A user's guide can be found in the doc subdirectory.

Let us look at some specific examples of how Orbiter can be used through the command line. The command

```
orbiter.out -v 3 -linear_group -PGL 4 7 -wedge -end \
  -group_theoretic_activities -surface_classify -end
```

classifies all cubic surfaces over the field $\mathbb{F}_7$ under the projective linear group. If desired, it is possible to use

```
orbiter.out -v 3 -linear_group -PGGL 4 4 -wedge -end \
  -group_theoretic_activities -surface_classify -end
```

to perform the same classification with respect to the collineation group $\mathrm{P\Gamma L}(4, 4)$. The `-surface_recognize` option can be used to identify a given surface in the list produced by the classification. For instance,

```
orbiter.out -v 3 -linear_group -PGGL 4 8 -wedge -end \
    -group_theoretic_activities -surface_recognize -q 8 \
    -by_coefficients "1,6,1,8,1,11,1,13,1,19" -end -end
```

identifies the surface

$$X_0^2X_3 + X_1^2X_2 + X_1X_2^2 + X_0X_3^2 + X_1X_2X_3 = 0 \tag{1}$$

in the classification of surfaces over the field $\mathbb{F}_8$. This means that an isomorphism from the given surface to the surface in the list is computed. Generators of the automorphism group of the given surface are computed as well, using the known generators for the automorphism group of the surface in the classification. For instance, executing the command above creates an isomorphism between the given surface and the surface in the catalogue. Orbiter can compute isomorphism between two given surfaces, as long as both surfaces have 27 lines. For instance, the command

```
orbiter.out -v 3 -linear_group -PGGL 4 8 -wedge -end \
    -group_theoretic_activities -surface_isomorphism_testing \
      -q 8 -by_coefficients \
        "5,5,5,8,5,9,5,10,5,11,5,12,4,14,4,15,1,18,1,19" -end \
      -q 8 -by_coefficients "1,6,1,8,1,11,1,13,1,19" -end
```

computes an isomorphism between the two $\mathbb{F}_8$-surfaces

$$\begin{aligned}0 &= \alpha^3X_0^2X_2 + \alpha^3X_1^2X_2 + \alpha^3X_1^2X_3 + \alpha^3X_0X_2^2 + \alpha^3X_1X_2^2 + \alpha^3X_2^2X_3\\ &\quad + \alpha^2X_1X_3^2 + \alpha^2X_2X_3^2 + X_0X_2X_3 + X_1X_2X_3,\\ 0 &= X_0^2X_3 + X_1^2X_2 + X_1X_2^2 + X_0X_3^2 + X_1X_2X_3.\end{aligned}$$

The isomorphism is given as a collineation:

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & 3 & 2 & 4 \end{bmatrix}_2 .$$

In here, the numerical representation of elements of $\mathbb{F}_8$ as integers $0, \ldots, 7$ is used. The subscript indicates the power of the Frobenius automorphism that is associated with the collineation.

A second algorithm to classify cubic surfaces is available in Orbiter also. For instance, the command

```
orbiter.out -v 3 -linear_group -PGL 4 13 -end \
    -group_theoretic_activities \
    -classify_surfaces_through_arcs_and_trihedral_pairs 13
```

classifies all cubic surfaces with 27 lines over the field $\mathbb{F}_{13}$ using this algorithm.

Besides classification, there are two further ways to create surfaces in Orbiter. The first is a built-in catalogue of cubic surfaces with 27 lines for small finite fields $\mathbb{F}_q$ (at the moment, $q \leq 101$ is required). The second is a way of creating members of known infinite families. Both are facilitated using the `-create_surface` option. For instance,

```
orbiter.out -v 3 -linear_group -PGL 4 13 -wedge -end \
    -group_theoretic_activities -create_surface \
    -family_S 3 -q 13 -end
```

creates the member of the Hilbert, Cohn-Vossen surface described in [4] with parameter $a = 3$ and $b = 1$ over the field $\mathbb{F}_{13}$. The command

```
orbiter.out -v 3 -linear_group -PGL 4 13 -wedge -end \
    -group_theoretic_activities -create_surface \
    -q 4 -catalogue 0 -end
```

creates the unique cubic surface with 27 lines over the field $\mathbb{F}_4$ which is stored under the index 0 in the catalogue. It is possible to apply a transformation to the surface. Suppose we are interested in the surface over $\mathbb{F}_8$ created in (1). The command

```
orbiter.out -v 3 -linear_group -PGGL 4 8 -wedge -end \
    -group_theoretic_activities \
    -create_surface -q 8 -catalogue 0 -end \
    -transform_inverse "1,4,4,0,6,0,0,0,6,2,0,1,7,0,4,0,0"
```

creates surface 0 over $\mathbb{F}_8$ and applies the inverse transformation to recover the surface whose equation was given in (1). The surface number 0 over $\mathbb{F}_8$ is created, and the given transformation is applied in inverse. The commands `-transform` and `-transform_inverse` accept the transformation matrix in row-major ordering, with the field automorphism as additional element. It is possible to give a sequence of transformations. In this case, the transformations are applied in the order in which the commands are given on the command line.

References

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