The `QuantifierElimination` Package in Maple for QE and Real Algebraic Geometry

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`QuantifierElimination` is a package for Maple in collaboration with Maplesoft. The intention is to be a toolset for solution of Quantifier Elimination over the Reals (QE) problems, which frequently occur in Real Algebraic Geometry. In some sense a subset of QE problems, Satisfiability Modulo Theory (SMT) problems over Non-linear Real Arithmetic or Linear Real Arithmetic (`QF_NRA` and `QF_LRA`, in the terminology of `SMTLIB` [1]) are also amenable to be solved by QE methodology. The main QE procedure within `QuantifierElimination`, `QuantifierEliminate` takes a developmental "poly-algorithmic" approach to QE [4], where the poly-algorithm is between Virtual Term Substitution (VTS) [5] and the well known Cylindrical Algebraic Decomposition (CAD) [2], which has other uses in algebraic geometry beyond QE. The former is usually "completed" by the latter due to the former's reliance on the input formula being of low degree, but the methodology here is different to that of existing implementations. CAD's performance suffers when traversing problems in many variables. It is an expectation that this new approach enables additional performance, especially of SMT by VTS into CAD.

Another aim of the package is "rich output". An example is "witnesses" for QE, which aim to prove the equivalence of a quantifier free output to the quantified input in eligible cases. In the case of SMT, a set of assignments of variables to real numbers may immediately prove that the input formula is satisfiable.

`QuantifierElimination` also includes a standalone instantiation of CAD, that may be used in a context beyond QE. A recent development is new features for standalone CAD such that one can query and interact with a produced CAD. Example features include queries on signs of polynomials on cells, and truth values of cells on the original input formula. Such features meet requests from the algebraic geometry community.

Lastly, a usual desire of such software is "incrementality". This involves modification of existing data structures to reflect a modified input problem. Usually, this means addition of clauses to a formula. `QuantifierElimination` supports this, for both VTS [3] and CAD, and even follows the poly-algorithmic method of `QuantifierEliminate`.

References

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