A variant of van Hoeij's algorithm for computing hypergeometric term solutions of holonomic recurrence equations

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Abstract. Let $\mathbb{K}$ be a field of characteristic zero. A general representation of a hypergeometric term can be written as

$$C^n \cdot R(n) \cdot h(n), \tag{1}$$

where $C \in \mathbb{K}$, $R(n) \in \mathbb{K}(n)$, and $h(n)$ is a hypergeometric term expressed in terms of factorials and Pochhammer symbols such that $h(n+1)/h(n) \in \mathbb{K}(n)$ is monic (see [3, Chapter 9] and [1]).
Given a holonomic recurrence equation

$$\sum_{i=0}^{d} P_i(n) \cdot a_{n+i} = 0, \tag{2}$$

for the indeterminate sequence a_n, and the coefficients $P_i(n) \in \mathbb{K}[n], i = 0, \ldots, d$, we compute a basis of the subspace of all hypergeometric term solutions of (2) with representations (1) efficiently as van Hoeij's algorithm [5] does, without explicit use of the Newton polygon algorithm or the computation of finite singularities. The latter is avoided because our algorithm builds $h(n)$ in (1) by considering monic factors $u(n)$ of $P_0(n)$ and $v(n)$ of $P_d(n)$ in (2) so that $u(n)/v(n) = h(n+1)/h(n)$. Such a consideration is valid thanks to Petkovšek's algorithm [4] whose influence in our method can also be seen in the way we compute $R(n)$ in (1) and local types (containing the values of C in (1)) at infinity of hypergeometric term solutions of (2).
For our implementations, we use the computer algebra system Maxima which does not have such an efficient implementation for solving holonomic recurrence equations. As an example, using the algorithm in [2], it can be shown that the Taylor coefficients of $\sqrt{1+z} + 1/\sqrt{1+z}$ satisfy the recurrence equation

$$4 \cdot (n+2) \cdot a_{n+2} + 6 \cdot (n+1) \cdot a_{n+1} + (2 \cdot n - 1) \cdot a_n = 0,$$

for which our Maxima implementation yields the following basis of hypergeometric term solutions

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(%i1) HypervanHoeij(4*(n+2)*a[n+2]+6*(n+1)*a[n+1]
      +(2*n-1)*a[n]=0,a[n]);
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$$(\%o1) \qquad \left\{ \frac{(n-1) \cdot (-1)^n \cdot (2 \cdot n)!}{(2 \cdot n - 1) \cdot 4^n \cdot n!^2} \right\},$$

which is equivalent with Maple's output obtained using *LREtools[hypergeomsols]*.

Keywords: Holonomic recurrence equations · Hypergeometric terms · van Hoeij's algorithm · Petkovšek's algorithm.

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